Foundations of Perceptual Theory
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ON THE NEED FOR A GENERAL QUANTITATIVE THEORY OF PATTERN SIMILARITY

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ABSTRACT

The concept of similarity evokes different meanings for different people. However similarity is conceived, it certainly plays an important role in the modeling of pattern perception. The contrasting approaches toward psychological similarity taken by the measurement theorists, the psychometricians of multidimensional scaling, and the process model oriented cognitive psychologists should be reconciled if a full account of the role of similarity in perception is to be adequately developed. This essay attempts to initiate a preliminary synthesis of these approaches while introducing some important philosophical questions concerning the application of multidimensional scaling (*MDS*) models of similarity. These questions arise from a deeper analytical investigation of the foundations of modeling perception. Along the way, the reader is introduced to mathematical concepts from topology and geometry that underlie the use of *MDS*. It is hoped that the discussion will provide some tutorial benefits as well as open the door to a more unified treatment of similarity and its role in psychological representation.

The general goals of the present anthology involve a search for reasons that a generally accepted theory of perception has not arisen after more than a century of concerted experimental and allied theoretical attack. As asserted in the Preface, this state of affairs calls for a determined examination of the conceptual and philosophical foundations of perceptual science. A second possible subsidiary aim is the analysis of the reductionistic case for perception. Before taking up the main purpose specific to this essay, a few comments are in order regarding the goals that are general to the book as a whole.

Just as in any complex question in science, there are probably a number of contributing causes to the absence of a general and accepted theory of perception. Some of these are likely remediable whereas others may be inherent to the field. Inherent difficulties involve some parts of the reductionistic problem. Scientists work at a myriad of different levels of perception, from macroscopic regions such as perceptual categorization and perception of emotional expression, to biochemistry and quantal effects at sensory synapses. The very languages of the different levels, not to mention the standard prejudices (Dr. Neuroscientist: "Research on macroscopic models and behavioral data is sloppy and unscientific." Dr. Macro-perceptual-scientist: "Neuroscience is usually so microscopic as to bear almost no relation to real perceptual phenomena.") militate against much in the way of interdisciplinary progress. Indeed the very plethora of levels under study suggest that a truly general, level-encompassing theory may be a chimera—impossible in principle. Quite obviously, even though chemistry can in principle be reduced to physics, only a tiny paucity of interconnecting laws have actually been formulated. This state of affairs is even more true of fields like organic chemistry, biochemistry, and microbiology.

Without wishing to give short shrift to the deep puzzles associated with the reductionist question, let us just make the following statement: Our own preference is that the behavioral or black-box theories not be in obvious contradistinction to the physiological or anatomical facts or "laws." However, outside of that mild prescription, we believe that it is legitimate and worthy to develop process models and theories, even so-called connectionist ones, without undue homage to the physiological world. This does not imply, of course, that provocative bridges cannot be built across behavioral and physiological disciplines.

Perhaps a more profound reason for the failure of a general perceptual theory to surface, even one at a single level of discourse, is a combination of: 1. Continually emerging challenging sets of data that upset contemporary theoretical "apple carts." 2. An inability of laboratories utilizing different theoretical and methodological tools to synthesize their findings, not to mention their perspectives, in order to produce a widely cogent theory. Thus, we find even at roughly the same level of attack, that some investigators focus on quite local phenomena, others quite broad phenomena. Perceptual scientists run the gamut with regard to their sophistication in mathematics, perceptual and/or psychophysical background and knowledge, and acquaintance with physiological lore, information processing vs Gestalt vs Ecological vs (insert your own favorite "school" of thought) factions.

The last problem brings up the particular subject matter of the present essay. A central hallmark of much research into several major areas of perception has been the concept of similarity. Such areas include many aspects of form perception (recognition, memory, discrimination, gestalt vs analytic, independent feature perception, etc.), categorization, study of perceptual dimension, perceptual scaling and measurement. The theme of similarity crops up increasingly in "higher processes" such as decision making and problem solving and thinking, and is considered to form a major axis of thought by a number of philosophers. Yet it seems fair to say that even a generally accepted theory of similarity, much less a general theory of perception, is far from reality. In searching for the reasons for this lacuna, we may hope to locate some of the causes of the analogous difficulty in more general theories. However, our foremost goal will concern similarity per se.

Certainly, even within this relatively limited arena, there are many directions a (somewhat philosophically bent) scientific sleuth might take. Our own investigation naturally centers around quantitative descriptions of the perceptual/similarity issue. In particular, it has struck us as extremely unfortunate that several different quantitative approaches, each of which has offered valuable information on the similarity qua perception topic, have not somehow been brought together, to the mutual benefit of each, and certainly to the perceptual community at large. We now move to discuss this specific issue and some of the many ramifications and enigmas it prompts.

The enormous amount of research accomplished on the psychological scaling of similarity roughly breaks down into three overlapping regions: the foundational approach, the substantive model based approach, and the descriptive multidimensional scaling approach (*MDS*).

This is not the place to attempt a detailed description of these approaches, or indeed even a thorough comparison, a task that might be valuable although challenging, having to our knowledge never been attempted. Also, the focus in this paper will be on relatively low-level pattern processes, rather than on higher level cognitive mechanisms (e.g., see van Leeuwen, 1992). For instance, it has long been evident that the brain is sufficiently agile to simulate digital or analog machinery, as the task demands. In any event, an added attraction (we hope) in the following account and discussion is a certain technical tutorial, but we feel valuable, background required for our deliberation.

In order to get things started, and at the risk of brevity in extremis: 1. The foundational approach examines the conditions on psychological scales treated as qualitative objects that are necessary and sufficient for their representation in various types of mathematically specified spaces. This approach has largely been confined to deterministic (non-probabilistic) spaces (e.g., see Krantz, Luce, Suppes, & Tversky, 1971; Luce, Krantz, Suppes, & Tversky, 1990; Suppes, Krantz, Luce, & Tversky, 1989). 2. The substantive modeling approach typically assumes some type of perceptual or psychological space, in many cases, endowed with probability distributions and then investigates predictions for various kinds of experimental paradigms. One example of this approach is the general recognition theory of Ashby, Townsend, and colleagues, which is a generalization of signal detection theory to the multidimensional case (Ashby & Townsend, 1986; Ashby & Perrin, 1988; Ashby & Gott, 1988; Ashby & Lee, 1991; Kadlec & Townsend, 1992a, 1992b). Other examples of multidimensional models of perception and similarity include the generalized context model of Nosofsky (Nosofsky, 1984, 1985, 1986) and the stochastic model of similarity and judgment of Ennis and colleagues (Ennis & Mullen, 1986a, 1986b; Ennis, Palen, & Mullen, 1988; Ennis & Mullen, 1992). 3. The descriptive *MDS* approach is most often applied to similarity ratings or other "proximity" data with the object of unveiling the underlying psychological space (e.g., Shepard, 1964a; Kruskal & Wish, 1978; Carroll & Arabie, 1980). The present study will touch on each of the three throughout the discussion. Although it is difficult, if not impossible, to be an expert in all these areas, we agree with Uttal (1988, 1992) and Newell (1991) that more theoretical and empirical syntheses are needed in cognitive sciences today.[1]

First, a disclaimer is in order. The similarity judgment format is intended only to provide a milieu for the raising of critical issues concerning psychological spaces, not to review data or come to theoretical conclusions about how it takes place. The concept of similarity judgment enables the putting aside of ancillary issues that arise in other perceptual and psychophysical contexts, such as identification, recognition, discrimination and so on. However, because some of these paradigms are currently popular in psychology, and because many of the most interesting theoretical work has been done in these contexts, we often have recourse to their approaches in our deliberation. It should be noted, though, that the concept of similarity can be very task dependent. Stimuli that are similar according to discriminability measures (e.g., same-different *RTs*) may not be rated as similar by subjects in a similarity judgment task. For example,

[1] We have attempted to deliver a set of references to readers that will aid them in broaching various literatures. Needless to say, in such a broad enterprise as the present, there is serious danger of omitting any particular reader's pet reference, not to mention one of their own important articles. We apologize in advance for any such transgression.

a common finding is that a tilted *T* is much easier to detect among a background of upright *T*'s than is an *L* among upright *T*'s (see Pomerantz, 1981, for a review). When asked to judge pairwise similarity, however, subjects rate the tilted *T* as more similar to upright *T*'s than the *L* is to upright *T*'s. If one accepts the hypothesis that ease of discrimination is inversely related to similarity these results appear to be contradictory. We believe the best approach to this problem is to develop a process model that systematically accounts for results of different paradigms.

A popular way of modeling similarity in psychology is to posit that percepts or other psychological "data" are represented as points (or probability distributions, if the system is noisy, as in Ennis, et. al., 1988) in an abstract psychological space and that (dis)similarities correspond to the distances between points in the space (see, Nosofsky, 1992, for a review). One reason, among others, for the success of these geometric models of similarity is the intuitive result of having a small number of dimensions with which to characterize a set of stimuli. In fact, part of deriving a scaling representation is to discover the smallest number of dimensions that, together with the distances between the points in the space, adequately capture the observed similarity orderings. Also, in some non-distance based models of similarity, such as the version of the aforementioned general recognition theory that applies to the similarity rating task (Ashby & Perrin, 1988; Ashby & Lee, 1991), psychological dimensions are still assumed to exist and play a central role in the derivation of similarity.[2]

The foundational approach too, typically assumes that some sort of primitive psychological "scales" exist to represent values on the equally primitive "dimensions," but the nice spatial properties are derived from weaker qualitative notions and machinery, rather than being assumed a priori, to exist unlike the other methods. For instance, in the *MDS* descriptive tradition as opposed to the foundational, one assumes to begin with, that, say, one of the members of the Minkowski family or "power" metrics applies, and uses one of the *MDS* algorithms to attempt to uncover which one and what the actual configuration is.

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[2] Some may be concerned about the applicability of geometric models of similarity due to demonstrations of violations of the basic metric axioms by Tversky (1977). However, it is the opinion of the authors that dimensional representations may be appropriate and many of the observed violations are perhaps due to stimulus or response biases entering into the decision process (e.g., see Nosofsky, 1991, 1992). Thus, a full process model needs to be specified so that contributions from each part of it may be assessed empirically. Furthermore, it is sometimes overlooked that many of the topological and algebraic conditions that underpin Tversky's feature model, are the same as those leading to dimensional representations. Only the relaxation of certain key links permits the abrogation of such predictions as the triangle inequality. Hence, many of the questions raised in the context of dimensional and metric representation surface again with regard to such models.

Within this context we set out the issues with which we shall deal. There will be lots of questions and few answers, but we feel that the discussion will help to open up important avenues of theoretical, methodological and empirical investigation. One of the central questions we will be concerned with is that of the origin and nature of the psychological dimensions. This topic necessitates some introduction to the concept of dimension as employed in mathematics, but also how dimensions may relate to stimuli on the one hand and mental representations, such as memories on the other.

Another issue concerns the implications that the "true" dynamic perceptual and cognitive transformations of these spatial representations have for the uncovering and modeling of the dimensions and spatial structure. Thus, it might be asked "why do *MDS* procedures seem to work as well as they do?" So far, there has been a paucity of research that establishes the *MDS* algorithms on a mathematically rigorous basis. Some cognitive psychologists, on the other hand, seem to feel that "if it works, why fix it." We discuss this aspect further below.

The foundational approach is certainly rigorous but faces similar queries. For instance, one may ask whether the kind of primitive assumptions that measurement theorists make about the psychological values actually mirror something of the essence of the way the brain acquires dimensional information about real-world information. How different are these assumptions and theorems from simply assuming the spaces a priori, and how are they helpful in our understanding? And yet further, are plausible psychological transformations on stimuli and later within the cognitive apparatus, likely to preserve or evoke the constraints required to produce the desired spaces? Thus, it may be asked whether such more primitive and weaker assumptions are more resistant to deformation than would be the tighter metric information, should one just postulate the latter. We shall discuss these and other issues, such as dimension or feature selection within a general topological framework.

To begin, consider the question as to whether the tenets that permit the establishment of a pleasing geometry are likely to be met by reasonable perceptual processes. To some this issue may seem to belong to the domain of psychophysics rather than cognitive psychology. In fact, some have argued that cognitive psychologists should not be concerned with the dependence of psychological attributes on physical properties but only with the structural relationships among the perceived objects in an abstract psychological space (e.g., see Shepard, 1982, 1987; Nosofsky, 1992). Nevertheless, we propose that it is the task and variation in the (physical) stimulus that together with psychological transformations produce dimensionality, the topology, and geometry (metrics, etc.) of this psychological space. Without denying the value of the psychological space per se, we tend toward the view of S. S. Stevens (1951), who claimed that one cannot understand perception without understanding the nature of the stimulus, presumably in relation to the perceptual response. We would extend this notion to include the experimental variation that results in the set of stimuli presented to an observer, as well as the stated task requirement for that observer.

In the simplest case, the situation is the same as that stipulated by the foundational measurement theorists, namely, that each psychological dimension will be some function of the corresponding physical one (e.g., see Beals, Krantz, & Tversky, 1968; Tversky & Krantz, 1970). In more complex cases, the psychological dimensions may be many-many functions of a number of physical dimensions, but they are functions of the physical characteristics nonetheless. Most investigators tacitly appear to accept this fact, while perhaps overtly denying to have interest of it, when they explicitly manipulate specified physical dimensions (such as size or orientation of angle) often orthogonally to obtain similarity data. At present, any investigation of the foundational issues of "cognitive scaling" must rely on the measurement approach which, in turn, predicates the above scenario.

Even if one accepts the attempt to disengage psychological and physical geometry, it remains to be justified theoretically why or whether, in fact, a real organic cognitive system needs to possess representations that are mathematically or analytically "nice" such as metric spaces. Ultimately, mental representations have to serve the needs of an organism responding to stimuli (i.e., physical stimuli) in the environment. Why and how a metric distance representation of similarity should arise to serve this need, and be preserved throughout the dynamic processing flow, remains to be explained. For instance, could it be that arguments in the vein of the evolutionarily optimal assumptions proposed by John Anderson (1990) might be advanced? Or, may cogent ideas based on postulates that cover very large classes of models (e.g., employing general structures such those based on geometric measures as in Shepard, 1987) or generation through concepts of ecological necessity, (e.g., Gibson, 1986; Burton & Turvey, 1990; Cutting, 1991) be put forth? Of course, although deductions following from the latter two approaches are certainly testable, it is not obvious that the more philosophical precepts under which they were initiated are falsifiable.

In order to fully comprehend the issue of perceptual similarity, one must understand the origin of the psychological space with respect to the environment in addition to any subsequent generalization processes that act on it. Now, it seems reasonable to conclude from the many multidimensional scaling studies, that mutually related psychological dimensions do in fact exist with, at least, ordinal properties. However, the existence and especially the uniqueness of particular metrics must impress the disinterested reader as much less well-founded. Even the claim that the city-block metric is (or should be) associated with separable stimulus dimensions has come under increasing doubt.

In our opinion, an ultimate goal should be the enfolding of all three of the above approaches to psychological similarity into the substantive modeling approach. In fact, our perspective throughout this paper will be to conceive of the overall cognizer as a dynamic system, which will ultimately serve as a foundation for other theoretical ventures. First we break the overall system into a small set of subsystems, whose basic responsibilities are time-honored. These are: Early transformation and, when needed, feature analysis, and then a later comparison or matching phase, followed finally by a possibly biased response. A difference from previous approaches is that we are beginning an investigation of what happens to dimensional structure and similarity structure throughout the processing chain. The present study therefore is also encouraging the institution of a synthesis of process models with measurement models and with topological and geometric questions. A fully temporally specified information processing model of similarity ratings must ultimately specify the actual dynamic of processing. Our discussion is intended to broach wider questions of measurement, topology and dimension within the context, but without the specifics, of dynamic information processing.

A word is in order concerning the ubiquitous effects of context, particularly on scaling results. Early research on psychophysical scaling centered around discovering "the" function, be it logarithmic, power, etc., relating subjective magnitude to physical magnitude or intensity. Lurking behind this body of research was the tacit assumption that early sensory and low level perceptual processes are relatively invariant across situations in their structure and mechanisms. This view has come under increasing doubt. A striking example of this failure of invariance can be found in a classic experiment by Garner (1954) in which, on each trial, he asked subjects to state whether a second tone was more or less than half as loud as a first tone presented at the beginning of the trial. However, only half of the subjects heard tones that were lying above as well as below the intensity level conventionally accepted as half as loud as the standard tones. The other subjects should have protested as they only heard tones below the accepted half-way point. In fact, they did not protest, which has been taken as evidence that context seems to have forced subjects to scale the stimuli differently than in a standard magnitude estimation paradigm (for implications of this finding see Laming, 1991, in his commentary of Krueger, 1989). More recently, Marks (1992) demonstrates another dramatic effect of stimulus context on subjective loudness perception. He showed that relative loudnesses of two tones of fixed but different frequencies changed reliably as a function of the intensity range comprising the stimulus set in both a scaling paradigm and a direct matching task. Other empirical phenomena in scaling and dimensional judgment literature such as the semantic congruity effect, stimulus density and frequency effects, among others, speak to the obvious need to include context in any full model of the perceptual process (Banks, 1977; Birnbaum, 1974, 1978; Çech & Shoben, 1985; Holyoak & Mah, 1981, 1982; Parducci, 1965, 1992; Petzold, 1992). In the present paper, however, we will have to overlook context effects in our treatise not because they are unimportant but simply to keep the discussion at a manageable level.

Although some topics from relatively abstract mathematics will be employed, considerable care will be taken to ensure that the reader with at least a background in calculus will be able to follow the line of discussion.[3]

TOPOLOGY, UNIFORM SPACES, AND DIMENSION

An important aspect of our discussion centers around foundational measurement work on psychological scales of distance and similarity. The background with which the foundational approach begins, includes an impressive history of mathematical work in geometry and related fields. We cannot hope even to adequately limn in this history and the reader is referred to Suppes, et. al., (1989; vol. 2) for a review and many additional references. However, it seems to us that to start to appreciate the excitement as well as the daunting challenges of the similarity scaling problem, a brief acquaintance with the some of the coarser spaces underlying geometry is invaluable. We want to note, in starting that some of the following topological notions can be supplanted in measurement structures by algebraic counterparts, although certain technical "analytic" conditions are always present.

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3 We are of the belief, now that experimental psychology journals are no longer terrified of some calculus and elementary probability theory (albeit often consigned to appendices), that the time is nigh where psychologists could, with profit, bring to bear on theoretical problems, more advanced avenues of mathematics (see Cliff, 1992, for related comments with respect to measurement theory). Some writers, including us on occasion, whose own research necessitates quite abstruse tools, attempt to avoid frightening off the reader by trying to reduce the discussion to common parlance. This strategy can be pedagogically valuable. However, we suspect that sometimes it may be preferable to begin to introduce readers to some of the common terms and concepts employed in such fields in order to help them begin to delve into them on their own.

Basically, a *topology* on a set of points X is a collection, τ, of subsets of X, called "open sets" with the following property: This collection is closed under finite set intersection and unlimited set union. In addition, the whole set, X, and the empty set, ϕ, are in τ (i.e., are members of the collection of open sets). Closed sets are defined as the complements of open sets in the sense that C is a closed subset of X if and only if there exists an open set O such that $C = X\text{-}O$. Special cases of open and closed sets are the open intervals and closed intervals of the real line, with its ordinary geometry. Strictly, a topological space is denoted by the tuple (X, τ) but often just called X with the topology assumed. Such spaces can be so ill-organized that little of interest can be said about them.

One can build topological spaces of increasing "strength" by imposing more and more constraints. After sufficient such conditions are imposed, the topological space is metrizable, that is, a metric $d(x_1, x_2)$ can be defined on all points x_1 and x_2 in X obeying the usual properties:

$$d(x_1, x_2) > 0 \text{ if } x_1 \neq x_2,$$
$$d(x_1, x_2) = 0 \text{ if } x_1 = x_2,$$
$$d(x_1, x_2) = d(x_2, x_1),$$
$$d(x_1, x_2) + d(x_2, x_3) \geq d(x_1, x_3).$$

Such a metric must deliver the same topology with which one began. That is, all the open sets of the original topology must be derivable from sets of the form, $B(x, d, r)$ where B is a "ball" or "sphere" around the point x, based on metric d, of "radius" r. Such sets form a so-called "basis" of the space and all open sets can be generated from these by taking unions and finite numbers of intersections. Of course, one can in many cases simply assume that a metric, for example, the Euclidean, applies to a space of points (in this case, with the usual set of orthogonal coordinates), and the topology generated immediately from that metric (see Munkres, 1976, for more details).

A very interesting type of space exists intermediate between "raw" topological spaces and metric spaces; these are known as "uniform spaces." These are a bit more abstract and the reader meeting them for the first time may not fully comprehend them. However, their importance in the present context can probably be appreciated. There are several equivalent ways of defining them. The one we use seems fairly reasonable. A uniform space, like a topological space, is based on a class of subsets. However, in this case the class of subsets U of a set X is a set of subsets from the Cartesian product XxX, that is, U is composed of pairs of points from X. Moreover, in this instance, the class is closed under finite intersection and inclusion, instead of finite intersection and arbitrary union. The inclusion implies that if set A is in the class, and A is a subset of set B, then B is in the class also. Sets such as A, B that are in U are typically called "ensembles" (alternatively some authors call these "entourages") and generically denoted by E. Also unlike topological spaces, the empty set ϕ is not included in the class, although the "universe" set, XxX, the set of all pairs of points from X, is included in U.

We do need the notion of "concatenation" of the ensemble sets, for example, $AB = C$, also a member of U. This concatenation builds another ensemble by forming, for all pairs (x_1, x_2) and (x_2, x_3) the pair (x_1, x_3). All such final pairs are in C.

In addition there are some additional conditions that a uniform space must meet: 1. The intersection of all ensembles in U is precisely the *diagonal*, D. That is, D equals the set of all pairs (x, x) where x is contained in X. 2. For any ensemble E, E^{-1} is also an ensemble, where E^{-1} is the set of pairs (x_2, x_1) where the pairs (x_1, x_2) are contained in E. Thus, E^{-1} is made up of the reverse pairs of E. 3. For any E in U there is another ensemble B such that $BB = E$ (for an introduction to uniform spaces and their relation to topological spaces see James, 1987).

Although the above description is indeed rather abstract, we can espy in them a primitive and qualitative premetric structure. To see this, suppose we take a metric space, X, with metric d, and view the resultant ensembles and their relationships. First, construct the ensembles by setting $E_\varepsilon = \{(x_1, x_2) \in XxX \mid d(x_1, x_2) < \varepsilon\}$, that is all pairs whose distance in X is less than ε go into the particular ensemble associated with the positive real number ε. Now, we can discover in (1) a primitive, nonnumerical version of the minimality of a metric, that $d(x, x) = 0$, in the fact that the diagonal [with points of the form (x, x)] is a subset of every ensemble, since (x, x) belongs to all such ensembles. Obviously, due to the symmetry of the metric d, Condition (2) is trivially satisfied. Hence, (2) is a precursor of the symmetry found in the metric formulation. For (3), consider any ensemble E_ε and form the new ensemble $E_{(\varepsilon/2)}$. By the triangle inequality, $E_{(\varepsilon/2)} E_{(\varepsilon/2)} \subset E_\varepsilon$ so Condition (3) is fulfilled. Hence, (3) is a qualitative version of the triangle inequality. Finally, the reader may check that the other properties, such as closure in supersets (all sets containing smaller ensembles are also ensembles), are also satisfied in the metric case.

Now it can be shown that every uniform space generates a topology but more interestingly, every uniform space permits something close to a metric to be defined on it, a so-called pseudo-metric. A pseudo-metric has all the properties of a metric except that $d(x_1, x_2)$ can be equal to 0 without $x_1 = x_2$. That is, one can find points that are not the same point, yet have zero distance between them. That could be a good thing in some psychological applications. Note too, that the powerful triangle inequality is in force. Thus, the qualitative structure inherent in a uniform space is formidable, and the original classes of ordered sets of pairs can be reproduced through the use of the numerical pseudo-metric. Even topological spaces that do not satisfy the conditions for a uniform space always permit a so-called semimetric to be imposed on them (see Kopperman, 1988). A semimetric has properties that are analogous to metrics but the "values" do not necessarily satisfy the stringent order relations of the real numbers.

Returning to uniform spaces, with some upgrading of the conditions, actual metrics can be brought forth, that reproduce the "uniformity" that is, the uniform space, just as a metric may reproduce the original topology. Foundational measurement investigators, in collaboration with a mathematician and drawing on classical qualitative geometric research in mathematics (e.g., Blumenthal, 1950; especially Buseman, 1955), employed the strong implications associated with the ordered classes of ensembles, to derive impressive metric space results in the 1960s (e.g., Beals & Krantz, 1967; Beals, et. al., 1968). The important point here, of course, is that the qualitative notions accompanying these primitive spaces, demand less in assumptions than simply assuming specific metric spaces. Or, more accurately, they tell what conditions must be obeyed in real world systems, in order for a metric to exist. Yet, if the psychological data obey the qualitative conditions (along with some untestable "technical" conditions"), then we may be assured that the implicated psychological metric space exists, and at least in principle, can even construct it.

Perhaps the most important conditions for psychologists, in addition to the uniform space assumptions, are those "betweenness strictures" that help to produce "geodesics" between any two points in the space. Geodesics are analogs to straight lines in Euclidean space and thus take on the character of distance in the more general spaces. Beals and Krantz (1967) show that their conditions allow the imposition of unique geodesics on their spaces, geodesics that are unique up to multiplication by a positive scale (much like the ratio scale concept). For a less technical account with psychological discussion see Beals, et. al., (1968). Further work applied knowledge gained in research on general foundational scaling issues, for instance, extensive, difference, and conjoint measurement, in order to specify conditions that were sufficient to define orthogonal coordinates and more specific and refined metrics (e.g., Tversky & Krantz, 1970).

The scale question plays an important role in such developments. Thus, suppose psychological distance only exists on an ordinal scale, then any transformation can be accomplished on the measurements that preserves the order of distance. But this leaves the uniform structure unimpaired, which is what we need. Conversely, the pseudo-metric that can be defined on any uniform space might be only ordinally unique, or might lie on a stronger scale, such as ratio. These questions deserve much more treatment, including the relationship of the ordinal metric or pseudo-metric to the scales of the individual object measurement. They shall have to be left aside for now. In addition to the works on measurement listed above, Townsend and Ashby (1984) cite dangers in ignoring questions of scale type and Townsend (1992c) applies measurement scale principles to reaction time. Townsend (1990b) gives results concerning a hierarchy of distributional relationships that are impervious to monotonic perturbation and hence are robust up to ordinal scale types.

We have been emphasizing premetric and metric structure in our brief description, but lurking in the background is the notion of dimension, a concept not always fully understood outside mathematics and a small group of practicing scientists. Often, we still may tend to think of dimension as simply the number of quantities or measurements that are necessary to specify something. There is some truth in this statement, but like many qualitative ruminations, danger lurks in the absence of a precise formulation. For instance, Cantor (1874, e.g., see Noether & Cavaillés, 1937) demonstrated a one-to-one correspondence between the unit interval and the unit square, seemingly implying that two dimensions could be made "equivalent" to one. However, this function was not continuous. Furthermore, Peano (1890) showed that it is possible to contrive a function that maps the unit interval in a continuous fashion, onto the entire unit square. Thus, in this mathematical sense, two numbers in the plane are like one number on the unit interval. But this function was not one-to-one. That is, two points in the square correspond to a single point on the line.

This kind of research eventually led to a rigorous definition of dimension, especially through the respective work of Brouwer, Menger, and Urysohn (e.g., see Hurewicz & Wallman, 1941). A rigorous discussion of topological dimension appears in the appendix of this essay. Informally, a space of dimension n can be covered with arbitrarily small "cubes" (generalized to the notion of n-sided cubes) in some manner so that no point of the space is contained in more than $n+1$ of the cubes; in addition, if the cubes are small enough, then at least $n+1$ of these cubes must have a point in common. Moreover, dimension in this sense is preserved by so-called "topological" functions. The latter are one-to-one, onto, and continuous in both directions, that is both $f(x)$ and $f^{-1}(x)$ are continuous. Such a function goes by the technical name of *homeomorphism*. Homeomorphisms always preserve topological properties, but not necessarily geometric properties. For instance, if a space is connected (i.e., cannot be de-composed into two nonintersecting open or closed sets), then a homeomorphism to another space must necessarily preserve this feature. However, it would not necessarily preserve distance between two points or, in vector spaces, the angle between two vectors.

A small intriguing "aside" that can be skipped on first reading: Interestingly, if we consider only the rational numbers on the unit interval, we no longer have one dimension, but zero dimension! This is one of the paradoxes (or more likely, "antinomies," an apparent paradox) of the application of mathematics to science. Even though the rational numbers suffice to describe any physical phenomenon (i.e., there are enough of them and they are close enough together), most of the mathematics modeling the real world cannot be produced by the rationals alone (at least at present). And, the concept of dimension plays a strong role in science; yet, the dimension of the rationals, as pointed out, is zero. It seems likely that limitations in the accuracy of real world measurement and possibly the presence of statistical error, imply that our natural laws, taken in reference to these "coarse" measurements, act as though they were actually on the real line; for all practical purposes. The sparseness of the rationals cannot make themselves felt in such circumstances. Various philosophical and scientific aspects regarding the application of mathematics to science are discussed in a recent paper by Narens and Luce (1992). Townsend and Kadlec (1990) discuss the use of mathematics in psychology.

How should stimulus patterns be defined? In many cases, a single stimulus can be specified by its values on physical dimensions. However, in even moderately complex patterns, there exist an infinity of ways in which it could be deformed to produce other patterns. It is in the potential regularity of variation in a sample of stimulus patterns that an observer must select dimensions or features to which to attend, or in any case, to employ in various psychological tasks. A fortiori, aspects of a set of stimuli that are not undergoing change are useless in assessing similarity, discrimination and so on. Consider for illustration an observer shown a graph of a continuous function $y = f(x)$, with say, x lying between 0 and 1. Now there are an uncountable set of points at which the experimenter could, in principle, perturb the function in order to produce a new one. She/he might alter the third derivative, add a sine wave etc. Only the variation can tell the observer what to use to perform the necessary task. Note also in this connection, that a set of potentially infinite dimensional objects can often be defined by a single dimension. Thus, the class of functions $y = f(x) = exp(-at)$ is defined by the single parameter a, hence by a single dimension, the latter having an infinite number of potential values.

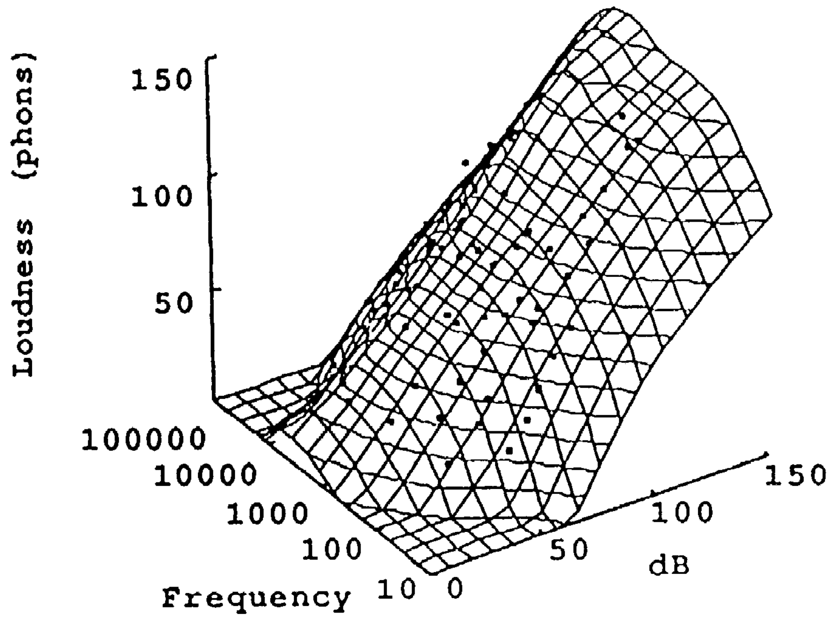

Figure 1. Subjective loudness surface as a function of tone frequency and intensity fitted using SYSTAT to the data of Robinson and Dadson (1956). The curves of constant height running from the back to the front of the surface are the equal loudness contours often projected onto the frequency-intensity plane when plotted as in Fletcher & Munson (1933) and Robinson & Dadson (1956).

Any finite set of stimuli probably can be produced from a finite number of dimensional variations. Of course, these might be impossible, from a practical standpoint to ascertain. However, in some cases, such as face perception, the capability of performance in certain tasks, such as same-different discrimination, is so capacious, as to suggest an almost infinite dimensional perceptual space. More on high dimensional perceptual spaces later.

Another important fact is that although the number of dimensions describing a space is invariant under a homeomorphic transformation, the way in which a space depends on the dimensions could be very different indeed. One such example is that of the auditory perception of a pure sinusoidal tone. As the name suggests, pure sinusoidal tones can be characterized as a sine-wave corresponding to the displacement of air particles as a function of time having the components of maximum amplitude (or intensity),[4] frequency, and phase angle. Ignoring phase,[5] a pure tone can be completely described physically by the amplitude of the wave and the frequency or cycles per unit time. The two major subjective attributes associated with pure tones are loudness and pitch. Hence, we have a two dimensional physical space being mapped into a two dimensional psychological space. However, loudness perception has been shown to be a function of both intensity and frequency (e.g., see Fletcher & Munson, 1933; Licklider, 1951; Stevens & Davis, 1938) thus violating a strong form of separability (see below). Figure 1, based on data from Robinson and Dadson (1956), shows the subjective loudness surface, measured in phons, plotted against intensity and frequency of the tone. The solid curves that run from back to front at a constant height are called *equal-loudness contours* and represent tones whose combination of frequency and intensity produce the same subjective loudness.

The fact that loudness depends on frequency as well as intensity can be seen in the curvature of these contours which indicate a maximal sensitivity with respect to loudness in the midrange of frequency for a given intensity level. In contrast to loudness, pitch perception, according to Licklider (1951), seems to depend primarily on the frequency of the tone (see also, Stevens, 1935; Morgan & Garner, 1947). Pitch, in non-musical contexts (see below), measured in mels, is a monotonic increasing function of tone frequency alone. Because subjective pitch, in mels, increases less and less rapidly as the stimulus frequency is increased linearly, it can be approximated, for our purposes, by a logarithmic function.

We can discuss this situation in more quantitative terms. Let the physical space, Φ, be composed of dimensions intensity and frequency denoted as X with values x and Y with values y, and the psychological space, Ψ, be composed of dimensions loudness and pitch denoted as U with values u and V with values v. We can describe the psychophysical transformation as a mapping, $\rho\colon \Phi \to \Psi$ where ρ is the vector function $(x, y) \to [f(x, y), g(y)] = (u, v)$. Here f is the function taking amplitude and frequency into subjective loudness (shown in figure) and g is monotonic function taking frequency into pitch. The requirements for ρ to be a homeomorphism is that it is one-to-one, onto, and bicontinuous. It is easy to see that ρ is onto since there are no points in auditory space that do not get hit by some combination of frequency and intensity. To verify one-to-one-ness (also known as injectiveness) is more difficult. Without knowing the exact analytical nature of the functions we offer the following qualitative argument. For

4 In this paper we will use amplitude or intensity as generic terms for pressure, power, energy, etc. However, in acoustics special meanings are invoked for these terms. For example, *intensity* always means energy flux density.

5 Phase information is also contained in the sine-wave representation. However, subjects can only discern this when presented with a reference (Licklider, 1951).

an injection, we have to show that given a particular intensity and frequency pair, (x_0, y_0), we have only one possible loudness and pitch pair, (u_0, v_0). This is equivalent to showing that the inverse function, $\rho^{-1}: \Psi \rightarrow \Phi$, is well defined. Suppose we have a loudness-pitch pair, (u_0, v_0). Because g is one-to-one and onto, we can find $g^{-1}(v_0) = y_0$, some frequency, uniquely. We combine this frequency value with the original loudness value, u_0, consult the equal loudness contours to determine the unique intensity level, x_0, that gives that loudness at that frequency. We can do this because at a fixed frequency, loudness is monotonically related to intensity. Hence, g^{-1} is well defined. From this we conclude that ρ is one-to-one. Clearly, ρ^{-1} is continuous since all function involved are bicontinuous. So, finally, if the qualitative forms of the various functions involved are correct, as indicated by the data, we may conclude that the function from intensity-frequency space to loudness-pitch space is a homeomorphism.

One might worry about the fact that the physical dimensions are actually unbounded whereas our ability to discriminate tones at the extreme ends of pitch and loudness disappears or is distorted greatly due to mechanical aspects of the ear. In this case, the transformation may not be a homeomorphism of all of frequency-intensity space because the set of tones that are audible may form a compact set (i.e., closed and bounded) in pitch-loudness space. Compactness is a topological property and, hence, is preserved by homeomorphisms. One way out would be to restrict the domain of the psychophysical mapping to be compact as well so that this restricted function could now be considered a homeomorphism. On the other hand, bounded sets may not necessarily be compact, particularly if they do not contain what are called limit points. The true psychological representation of tones may be immeasurable at the extreme ends of loudness and pitch due to experimental error, mechanical distortion, nonlinearities, etc. Characterizing the psychological space as bounded but open may be appropriate in this case. Hence, the transformation may still be a homeomorphism.[6]

In an earlier technical report, Zagorski (1973) attempted to test the hypothesis that ρ, the mapping from frequency-intensity space to pitch-loudness space, is, in fact, a homeomorphism in conjunction with the hypothesis that the metric that represented stimulus similarity for pure tones was *isotonic* in form. Isotone metrics are of the form $l[m(\Delta x), n(\Delta y)]$,

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[6] A concrete example to illustrate a homeomorphic mapping of an unbounded set into a bounded but open set can be seen in the fact that the whole real line can be mapped, homeomorphically, onto the open interval from -1 to 1, (-1, 1). The appropriate map $h: \Re \rightarrow (-1, 1)$ is given by $h(x) = 2x / [1 + (1+4x^2)^{1/2}]$. The inverse function is given by $h^{-1} = y / (1-y^2)$. One can check that this map is one-to-one, onto, and bicontinuous.

where l, m, and n, are strictly monotonic (isotone) functions (l in both arguments m and n) and Δx is the difference between stimuli on dimension X and Δy is the difference on dimension Y. A special case of isotonic metric is the popular Minkowski metric. He proved that if the psychophysical transformation is a homeomorphism and the metric describing similarity on the perceptual space is isotonic then the set of (two-dimensional) stimuli that are judged equally dissimilar from two reference stimuli must form an arc when plotted in the physical space. In a multidimensional bisection task, subjects were given two reference tones and then were presented with a test tone in which they could manipulate either frequency or amplitude (but not both) so as to make the test tone equally dissimilar to the previously presented reference tones.

Zagorski found that, when plotted in frequency-intensity (amplitude) space, the set of tones judged equally dissimilar did not form an arc but rather formed a criss-cross pattern. From this, he argued that an isotonic metric, in particular a Minkowski metric, was inappropriate to describe the representation of stimulus similarity. Subjects appeared to utilize either pitch or loudness but did not combine them on any given trial. It may be, however, that the method of responding induced the subjects to selectively attend to one of the dimensions given the fact that they could only manipulate one of the physical characteristics of the stimuli in the bisection task. Other evidence exists indicating that subjects do combine information from both dimensions when making decisions (Zagorski, 1973, 1975, 1978; Garner, 1974). In any case, the evidence does not convincingly rule out that the psychophysical transformation from frequency-intensity space to pitch-loudness space is a homeomorphism but seems to be relevant to the issue of separability versus integrality of the two dimensions. Later, we will return to the issue of perceptual or decisional separability and investigate how these concepts are related to our framework.

Certainly, not every transformation in which the perceptual system engages is a homeomorphism. One such example is the case of musical pitch perception in subjects who are familiar with the diatonic scale (e.g., Shepard, 1965; Deutsch, 1973). The relevant physical dimension is the tone's frequency. Now, however, the perceiving is done in a musical context. Consider the notes of a piano keyboard: *A*, *B*, *C*, *D*, *E*, *F*, and *G* which repeat themselves. Perceptually, notes of the same letter (i.e., an octave apart) are more similar than notes less than an octave apart. An octave interval represents a doubling in the frequency of the tone. This empirical result, known as *octave generalization* has led many investigators to argue that there are two dimensions of tonal perception, tone chroma, which corresponds to pitch of a tone within an octave, and tone height, which represents overall pitch (Meyer, 1904, 1914; Révés, 1913;

Ruckmick, 1929; Bachem, 1948; Shepard, 1964b). This phenomenon of octave generalization was the basis for the representation of the notes on the scale as a one dimensional helix *embedded* in a three dimensional Euclidean space, as shown in Figure 2 (see Shepard, 1982, for a historical review).

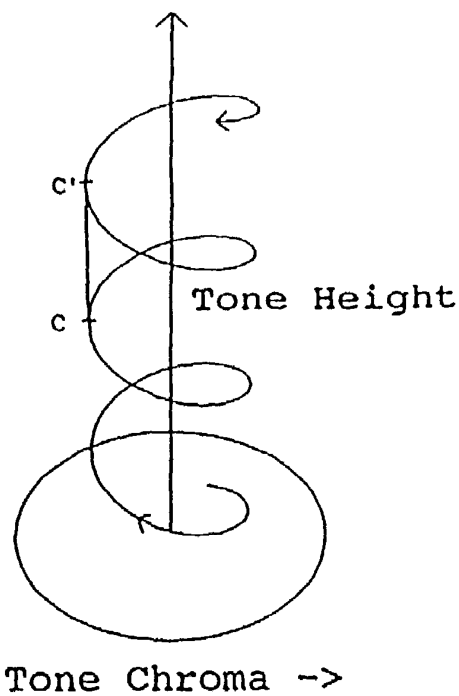

Figure 2. The helical model of musical pitch (as in Shepard, 1965). The two dimensions of musical pitch are termed tone height, representing the overall pitch level, and tone chroma, denoting within octave ordering of pitch. Pitches differing by an integral multiple of an octave (i.e., having the same name) line up vertically so that the Euclidean distance is minimized reflecting their high similarity.

A *topological embedding* is a map with certain properties. Suppose that f: $X \rightarrow \Psi$ is one-to-one where X and Y are topological spaces. Now the image set $Z = f(X)$ is a subspace of Y. That is, Z can be considered as a space which inherits its topology from Y. More precisely, the open sets in Z are the open sets in Y intersected with Z. In this example, X is the one dimensional frequency space which is essentially the positive real line, Z

corresponds to the one dimensional helix representing musical pitch and Y would be three dimensional Euclidean space in which the helix lies. The property that makes the map f an embedding is that, when restricted to the subspace Z, f is a homeomorphism, thus preserving dimension in the subspace. An interesting conundrum occurs if we note how the empirical observation that tones differing by exactly an octave are judged more similar than other pairs of nonidentical tones is being predicted by this representational scheme. It is the Euclidean distance in the embedding space, $\Re^3$, that is calculated to arrive at similarity and not arc length along the helical curve, the embedded space. This seems to be problematic when viewed philosophically. In the figure above, let the length of the straight line segment connecting the note C to the note C′ that is an octave away be the Euclidean distance representing dissimilarity of the two tones. Notice that no other points on this line segment besides the endpoints are realizable as possible tone perceptions. This means that, if this model is correct, the perceptual system has to traverse through something like a "no-man's land" where no tones are representable in order to evaluate similarities. Would it be preferable to develop an alternative geometric representation, or surface, that could capture the same distance orderings as this helical model while requiring distance calculations to be made on the surface itself? It is unclear at this time whether a (continuous) mapping from the above scheme to such a representation exists. More elaborate geometric characterizations of musical pitch perception capturing other similarity relationships, such as thirds, fifths, etc., which, however, meets up with similar questions, can be found in Shepard (1982) and Krumhansl & Kessler (1982).

Perhaps the best known example in perception is that of color, where hue, deriving primarily from the linearly ordered electromagnetic spectrum, curves around on itself under the transformation imposed by the visual system. First, if the psychological hue curve is conceived as topologically closed (i.e., contains all its boundary points), then the related transformation cannot be a homeomorphism because the part of the spectrum that is mapped into this curve seems to be an open interval (although possibly, we could artificially define two absolute thresholds at the top and bottom of the visible spectrum as boundaries of the visible spectral set). This is because whether a set is closed or open is a so-called topological invariant, that is, is preserved under homeomorphism.

More critically with regard to the present discussion about dimension, if the eye were like the ear with regard to wavelength, it would be infinite dimensional, because within just-noticeable-differences, every wavelength can be heard at arbitrary intensities. Of course, we only see combinations of the primary hues (see Krantz, Luce, Suppes, & Tversky, 1989, ch. 15, for more on this topic). So, in the case of hue there is a dimensional reduction as well as a curved transformation occurring.

The present writers suspect that many more phenomena in psychology can be described in terms of topological and geometric structures, but these may turn out to be more general than the ubiquitous Euclidean and Minkowski metrics in scaling methodology.

THE ENTRANCE OF DYNAMIC SYSTEMS THEORY

At this point, we need to discuss some aspects of dynamic systems theory. Our general perspective assumes that there exist three basic subsystems that are arranged sequentially, X, Y, Z, and that are responsible for an early transformation, a match or comparison operation, and a response bias and response selection operation. Although these are arranged sequentially and the immediate information flow is "upward" from X to Y etc., they can operate simultaneously, just as do most dynamic systems (e.g., Luenberger, 1979; Beltrami, 1987; for some special cases in elementary cognition, see Ashby, 1982; Bingham, 1991a, 1991b; Busemeyer & Townsend, 1989, 1992; Gregson, 1988; McClelland, 1979; Townsend & Ashby, 1983; Townsend & Busemeyer, 1989; Townsend, 1992a, 1992b; and in animal learning, see Killeen, 1992). Further, in general we believe there can exist feedback and top-down flow among these subsystems, but the present "barebones" discussion will have to ignore that aspect. Obviously, given that some neuroscientists believe there to be at least sixty known subsystems in the brain serving visual functions, the present scenario is surely a gross approximation, but we must start somewhere.

We write the state transition map for the first subsystem, X, as f: $T \times T \times X \times U \rightarrow X$. The state of X, which we call x, becomes the input to Y and so on. The function f tells how a state $x(t_0)$ at starting time t_0 gets transformed through time and with input $u(t)$, $t_0 < t < t_1$ to a new state $x(t_1)$. U is the set of acceptable inputs, X denotes the set of potential states, and T the time index. Hence, we could write $x(t_1) = f\,[t_0, t_1, x(t_0), u]$ in functional notation where f is our *state transition function*. Actually, in many cases of interest, the system is time invariant, which means that the starting point of the system is not critical, only the time interval over which the input persists. This allows us to simplify the notation to $x(t) = f[t, x(0), u]$. The next function, g, acts on $x(t)$, the value in X, over the same time interval, which of course, causes a transition of its own state over that interval: $y(t_1) = g[t, y(0), x]$. Note that although y can depend on x over the entire interval of time, all its information about the original input comes from x, not from the input u, directly.

We will concentrate on the first two subsystems in this paper. The presence of decisional or other biases, arising in the third subsystem, and their potential distorting influence on multidimensional scaling procedures has been recognized for some time (Townsend, 1971; Holman, 1979; Nosofsky, 1991). For instance, bias could mask underlying symmetry in a metric or perceptual similarity match, or even distort the minimality assumption, that the similarity between anything and itself is minimal, and the same for all objects. Nevertheless, here we wish to examine what might be going on early in the processing chain, with regard to dimensional and metric (and premetric) structure; namely in subsystems X and Y. Another constraint is that in this paper, we shall neglect other extremely important perceptual and elementary cognitive tasks such as low accuracy recognition, identification and confusion experiments, and categorization. The ultimate application of the ideas expressed in this paper to accuracy-confusion frequencies, reaction time and other dependent variables, in such milieus will be undertaken in further investigations. At that point, links will be drawn with general recognition theory (e.g., Ashby & Townsend, 1986; Ashby, 1989; Kadlec & Townsend, 1992a, 1992b) and other theories of identification and categorization (e.g., Nosofsky, 1986; Kruschke, 1992; Massaro, 1987).

What can happen in the first "stage" of processing? Certainly there is much loss of information due to limitations in our sensory organs, processing capacity and the like. In general, the function must be many-many in the sense that several distinct inputs can produce the same internal percept (e.g., Cutting, 1991; for an opposed view, see Burton & Turvey, 1990) and a single input may have the potential for producing a number of different perceptual results (e.g., as espoused in Bennett, Hoffman, & Prakash, 1989). However, in order to focus on what are for present purposes, the most important factors, let us assume that we have pared down the stimulus descriptions so that it is feasible for the observer's sensory apparatus to map the input in a one-one fashion to some central location. There will be ample opportunity in the future to relax such assumptions.

Let us consider, moreover, the relatively nice case in which the mapping is homeomorphic (see above), not only one-one but onto some image set of points, and bicontinuous. In some senses, all an observer needs in interacting with the real world is a homeomorphic map. Except in special cases, where she/he might also need velocity information, such a "smooth" and point-to-point correspondence should allow trouble free interactions. Indeed, it may be difficult to prove that higher order information, such as angle identicality or isometry (distance preservation), actually occurs in most cases. Of course, we are all aware of the many illusions, nonlinearities and probable necessity for a role of learning. Nevertheless, many theorists and experimentalists behave as if information is processed through the system in a relatively uniform fashion.

In any event, we shall see that even what appear to be minimal requirements for premetric information, can easily be disrupted. Thus, homeomorphisms, as would be expected, distort distances in general. For instance, even the innocuous function $y = f(x) = ax$ where $a \neq 1$ clearly alters distances. The order of distances of numbers is not, however, changed. But, in general, homeomorphisms also damage the order of distances.

A yet stronger type of transformation also is still too weak, one that is also one-one and onto, but is further uniformly continuous in both directions. In order to avoid long strings of conditions, let us agree to call such a function a "uniomorphism." Ordinary continuity implies that for any point x in X, say, one can find a small region around it so that any point x' in the region is as close as one likes to $f(x)$ in the range space, Y. Note that the size of the region around x can depend on where x lies. On the other hand, if a function is uniformly continuous, then, for a given disparity in the range space, Y, one can impose a region size in X so the regardless of where x lies in X, we are guaranteed that when x' lies in any such region, $f(x')$ will be in that close proximity to $f(x)$. Thus, uniform continuity is stronger than continuity. It turns out that uniform continuity going back and forth between two uniform spaces, preserves much of interest in the two spaces, and thus the linking of the names for the spaces and the transformations. Nevertheless, even a uniomorphism can, as claimed, change the order of distances. This can be seen in the simple example of $y = f(x) = x^2$ where x is confined to lying between 0 and 1 with the latter points included in the domain. Let $f^{-1}(y) = +y^{1/2}$. These functions are uniformly continuous on the closed unit interval and are one-one and onto; thus fulfill the definition of a uniomorphism. It can be seen that in the domain, 1/4 is farther from 0 than 4/5 is from 1, the respective distances being 1/4 and 1/5. After the squaring transformation, the respective distances of the separate pairs of points are 1/16 vs 9/25, the latter clearly larger, reversing the order of the distances.

Indeed, such transformations can even distort a premetric order of distances in the primitive uniform space underlying a metric structure. This follows logically from the above example, since the ordinary Euclidean metric imposes a uniform structure on the unit interval and the order of inclusion of the ensembles in the domain space is therefore violated after the transformation.

Thus, what we need is a transformation that is weaker than an isometry (preserving the actual distances) and stronger in the proper sense, than a uniomorphism. We can think of an ensemble in a uniform space as encompassing all points with a disparity less than some non-numerical level. Ensembles that contain the previous one, include even larger disparities and so on. Further, we may consider a transformation that preserves disparities as one that preserves the order of ensembles. Basically, we can state the definition as follows: A transformation between two uniform spaces that is one-one and onto, preserves disparities in both directions, is one in which the order of inclusion of the ensembles is preserved. That is, if f is the transformation $f\colon X \to Y$, and is one-one and onto then f preserves disparities if and only if whenever $E_1 \subset E_2$ in X, then $F_1 \subset F_2$ in Y, where F_1 contains all the pairs of points mapped from E_1 and none others and similarly for F_2. More specifically, if (x_1, x_2) contained in E_1 implies that (x_1, x_2) is contained in E_2, then $[f(x_1), f(x_2)]$ is in F_1 by definition and is required to also be in F_2. We make the same demands in the inverse mapping from Y to X.

An example of a distance order-distorting transformation can be found in the modeling of the identification and categorization relationship by Nosofsky (e.g., 1986, 1987). In identification tasks, the subject must produce a unique response when presented with a given stimulus. That is, identification requires a one-to-one map from stimuli to responses. In categorization tasks, the subjects assigns one response to several stimuli, a different response to other stimuli and so on. Here, we have a many-to-one mapping of stimuli to responses. Nosofsky has developed a model, which is a generalization of Medin and Schaffer's (1978) context model, that serves as a bridge between the two tasks. With this model, he has successfully predicted categorization performance based on knowledge of identification performance. The procedure he adopts is to first fit the identification data using Shepard's (1957; see also Luce, 1963) MDS-choice model. This step yields a multidimensional psychological space that captures the underlying "similarity" relationships between stimuli as used in the choice function contained in the model. The basic tenets of multidimensional scaling hold in that the similarity parameters of the choice model are assumed to be inversely monotonically related to the distance between the respective stimuli in the psychological space. Using this derived representation, together with the hypothesis that subjects distribute their attention appropriately in order to maximize performance, Nosofsky is able to predict subsequent categorization performance. The attention-optimization hypothesis can be realized in, for example, a categorization situation in which only one of, say, two dimensions are relevant for the task. Subjects are assumed to distribute attention in a way which is equivalent to stretching the space along the relevant dimension(s) and shrinking it along irrelevant ones and then recalculating distances between stimuli.

It is clear that pairwise distance orderings will, in general, be distorted as a result of the attention transformation. In fact, it is necessary for this to occur in the model if it is to successfully predict categorization data. Presumably, if the new distorted representation is stable, obtaining similarity judgments, post categorization, should reveal these new similarity orderings. Such experiments have been done, but with somewhat ambiguous results (Shin & Nosofsky, 1992).

CASE STUDIES: EXAMPLES OF SPACES UNDER PSYCHOLOGICAL TRANSFORMATION

Disquiet of some 18th century mathematicians concerning the God-given quality of Euclidean axioms and the absolute essence of Newton's incorporation of such spaces in his Principia, gave way in the 19th and early 20th centuries to a wholesale revolution. It became understood that any particular set of geometric axioms was not necessarily better in general than another. From the mathematicians point of view, classes of geometries are like a room full of games with somewhat different rules to play by. From the empirical scientist's perspective, a geometry (for that matter a topology etc.), is simply a model of a specific piece of reality that has to be empirically tested. Einstein was the first to really seize the opportunity to implement non-Euclidean geometries in science, although the mathematical foundation for his exploits had been laid by mathematicians such as Gauss, Lobachevski, Bolyai, and Riemann. Interestingly, Minkowski, an eminent mathematician of his day, was reportedly rueful that he failed to preempt Einstein in the application of Riemannian geometry to physics. In psychology, there is a substantial tradition of research attempting to specify the geometry most appropriate for human vision (e.g., see the review in Suppes, et. al., 1989, particularly discussing the work of Blank, Luneburg, and more recently, Foley and Indow).

In any event, the outside world, whether it is the visual, olfactory, auditory, or tactile (not to mention the internal sensory world of kinaesthesis etc.), does not necessarily have an absolute spatial description, when taken in reference to psychological phenomena. It has often proven convenient to start with standard physical descriptions of the stimulus patterns, when they exist, but the perceptual concomitants of these should not be taken to be equivalent, or even easily related, without demonstration. Moreover, it seems quite likely that the standard dimensions of classical physics, length, mass, and time were chosen over historical time because they are perceptually salient to human observers and more or less perceptually orthogonal, in their effects.

In the paradigm under discussion, the observer is presented with two objects and asked to rate their similarity or dissimilarity. Thus, they must both be transformed into an internal pattern or code and then matched.

Case I. A Finite Dimensional Euclidean Space

Consider an ordinary run-of-the-mill Euclidean space, one where the external dimensions are representable as a finite, orthogonal vector space endowed with the Euclidean metric, and it is invariant over time. For notational convenience, suppose it is three-dimensional, although this is entirely arbitrary. Thus, $u(t) = [u_1(t), u_2(t), u_3(t)] = (u_1, u_2, u_3)$. Assume that we consider the asymptotic stable percept $lim\ x(t) = lim\ f[t, u, x(t_0)]$.

Even if f maps to another three-dimensional Euclidean space, the dimensions could be complicated functions of the external dimensions, for example, $x(\infty) = f[\infty, u, x(0)] = [u_2\ log(u_1), sin(u_2)\text{-}sec(u_1), (u_3)^{1/2}]$. Thus, dimensional structure may be disfigured even though the number of dimensions is preserved. When will the order of distances be distorted? In most cases, from one perspective. A sufficient condition to preserve distance orders is of course, an isometry, which is defined as distance preserving. Note that even an isometry can distort the visual picture of a form. For instance, a sheet of paper can be wrapped into a cylinder while leaving distances unchanged.

However, a more interesting sufficient condition is to multiply each object (i.e., vector) in the space by a positive number. This alters Euclidean distance by the same number and leaves the order of distances unchanged. As observed earlier, the mild generalization of this concept to multiplying (i.e., stretching or shrinking) the different dimensions by different positive numbers, fails to leave the order of distances unchanged. If such linear transformations cause havoc with the distance orders, it is obvious that non-linear transformations would generally do the same. Readers may convince themselves that even letting the function of each dimension in Y be a strict monotonic function of a respective dimension in X and further requiring that all these functions be the same, will not generally preserve distance order. A generalization of the earlier one dimensional example is found by letting $y_1 = (x_1)^2$, $y_2 = (x_2)^2$, and $y_3 = (x_3)^2$; note that the dimensions are stretched in the same nonlinear way, but distance orders are not preserved.

These kinds of transformation can be given a higher sense of possible reality by hypothesizing that, say, a generalized Poisson counting rate $v_i(t)$ is a monotonic function of the dimension i value, for $i = 1, 2, 3$. It is thought that firing frequency of neurons may be an important coding principle on the part of the brain. If $v_i(t)$ approaches a stable limit as t becomes large, then we can form instantiations of the above possibilities. If, say, $v_1(t)$ is affected by dimension 2 as well as dimension 1 then we might write $v_1(t; x_2)$. If the limits are functions only of their respective external counterparts and are linear functions of the input values then $lim\, v_1(t) = k_1\, x_1$ and so on.

Next comes the operation of comparison or matching, which as we noted earlier, involves the mapping of the two psychologically specified three-dimensional depictions into the real numbers, usually the positive reals. Both the depictions and the result of the mapping may be non-numerical objects. For instance, the rating could just lie on an ordinal scale leading back to the notion of the uniform space concept. In any case, we will assume a numerical representation, since at some point the observer has to respond with a number name. (But be aware that this does not imply ratio scale properties!) How is this comparison made? Will the result bear any relationship to a metric? If, say, a similarity rating *is* monotonically related to psychological distance, where could one find that out; for instance, how could one ascertain the satisfaction of the triangle inequality? In fact, it is hard in general to do that without knowing the specific function that relates a similarity rating to underlying distance.

On the other hand, one should be able to test the proposition that nothing is more similar to an object than itself [akin to the diagonal being contained in every ensemble in uniform spaces and $d(x, y) = 0$ implies $x = y$, in metric spaces]. And, the same goes for the proposition that *all* objects are equally similar to themselves [forming an equivalence class of object pairs on the diagonal in uniform spaces and $d(x, x) = 0$ for all x in metric spaces]. Whether one can test for symmetry, depends on giving an interpretation to the order of the objects in $d(x, y)$ and $d(y, x)$. In some contexts, this makes sense, in others it may not.

What about something like a cross-correlation, $\Sigma\, x_i\, y_i$, a well known statistical procedure? In the pattern recognition and mathematical communications theory literature, it is shown that this procedure forms a so-called matched filter in certain cases (e.g., see Townsend & Landon, 1983), and when any added noise is white (zero correlation from one instant to the next) and Gaussian, then such a filter maximizes the signal-to-noise ratio. Furthermore, under more stringent conditions on the Gaussian distributions, it can be shown to deliver a minimum distance classifier. Of course, in such cases, one is dealing with the situation where a pattern is input to a classifier rather than with the context of matching two patterns for similarity or dissimilarity. The present situation precludes those conclusions.

For a given $\Sigma (x_i^2 + y_i^2)$, we do see an inverse monotonic relation between the cross product and the Euclidean distance. But, observe that the similarity of a pattern to itself, being constant across patterns, is not found with this measure, nor even that the similarity of a pattern to itself will be larger than non-identical patterns. It may be possible through some sort of normalization strategy in specific cases, to reinstate such properties, but it should be clear that they will not come about automatically.

Case II. Forms as Manifolds: Infinite Dimensional Pattern Spaces

Case IIA. Resting Manifolds. At this point we must outline what a manifold is. For more rigorous definitions, we refer the reader to standard works such as Boothby (1986), Auslander & MacKenzie (1977), or Spivak (1979), and for lots of high level intuition, Koenderink (1990). Actually, some of the intuitive backdrop to manifold theory can be garnered from studying surfaces in 3-dimensional Euclidean space and an excellent beginning text there is O'Neill (1965). Within or close to psychology we again recommend Koenderink (1990), and for more advanced applications from differential geometry, Hoffman (1966) and Lappin (1990). Recently, progress has been made on the contribution to three dimensional stereopsis to form perception (e.g., Norman & Todd, 1992) and also the influence of motion (e.g., Norman & Lappin, 1992) both within the context of geometries. However, the applications of such topics to psychology, have hardly begun to be tapped, and we predict that the rich knowledge base residing within topology and geometry will provide powerful tools for important segments of psychology for years to come.

A *manifold* is a space that can locally be mapped into an ordinary n-dimensional Euclidean space. That is, for every point in the space, there is a region, known as a neighborhood, around it, with the property that this neighborhood can be transformed in a smooth fashion (i.e., continuously to and from) to a neighborhood in Euclidean space. This is done carefully for every region in the original manifold with the edges of the neighborhood appropriately pasted (mathematically) together, and with consistency of the transformation for overlapping regions. The result is that operations on the original, possibly very abstract, space can be much more readily accomplished in the well-understood Euclidean space. Perhaps the simplest, not totally trivial, manifold is a two dimensional surface in three dimensional Euclidean space. Consider a very elementary example where a surface can be represented as a function that maps part of the Euclidean plane into a curved "roof" of some sort lying above the plane. This, of course, can be supplemented by a similar function carrying points on the plane to a "floor" below the plane and then connecting up the floor with the roof to make a closed, "empty" form.

Now suppose our set of stimuli is composed of such forms with different functions carrying parts of the plane to the surfaces. And let us assume that an internal homeomorphic representation exists for these forms. We set aside momentarily the issue of dynamics. Even if the map from external to internal representation is an isometry (distances are preserved) of the forms themselves, the question still arises as to the appropriate topology and/or geometry for this particular set of stimuli. Thus, the properties of the space would differ when one pattern is a balloon among a set of other toys as opposed to different types of elliptical objects, such as other balloons, a class of objects filled with air—but not necessarily balloons and so on.

Furthermore, in general we shall have to think of a "filled-in" space with more points than the present finite set of objects. In order to employ the classical and also powerful recent topological and measurement tools, the space must be sufficiently rich. This concept was touched on earlier in the chapter and can be vividly seen in the technical axioms required by the foundational measurement investigators to establish their results. When all the points of a form can be important in a task, or are simply employed by an observer for comparison or an analogous task, we have to construct an infinite dimensional space, and think about metrics and similar concepts on such spaces.

In some special cases, it may be possible to compare forms by constructing metrics or weaker comparisons on the functions that define them. For instance, consider the space of all continuous functions on the closed unit interval, $[0, 1]$, mapping it onto another unit interval. As we wrote above, $f: [0, 1] \rightarrow [0, 1]$. There are many metrics that could be used. One common metric on two such functions is $d(f, g) = \int [f(x) - g(x)]^2 \, dx$, that is, the integrated, squared distance of the functions from one another. Another is $d(f, g) = max\{[f(x) - g(x)]^2\}$, (the absolute value would also work in either case). In spaces where the functions or the difference may be bounded but that bound may not be attained by the functions, we may substitute "lub" (least upper bound) for "max." In any event, the first metric takes the distance of all points into account whereas the second depends only on the largest distance. Interestingly, the topologies generated by these metrics are not comparable. This fact implies that important topological notions that are even more general than metric ones, may be very different in the two spaces, even though the objects, that is the class of functions, are the same. Observe that this space of continuous functions on the unit interval is infinite dimensional, even uncountable (like the irrational number on the line, but even bigger). This is because there are an uncountable number of points along the interval $[0, 1]$ where the two functions can differ.

Infinite dimensional spaces immediately run into problems that do not encumber finite dimensional spaces; for instance, in many cases, no metric may exist at all on them. Moreover, some of the key concepts, such as a set of unique geodesics between points (remember, in more general spaces, these are the equivalents to straight lines in Euclidean space, and determine the general distance in small regions in the more general spaces), may no longer hold in such spaces. Thus, there is no unique "path" of minimal distance carrying us from one specified function f, to another g, in our current example.

This is an important consideration when we consider general shape perception questions, in high dimensional spaces. For instance, in pitch discrimination or face discrimination tasks (e.g., same-different tasks), the human sensorium is so high dimensional, it can probably be considered infinite dimensional as a good approximation to reality. Or, with regard to the question of geodesics, what is the shortest path of faces leading from one face to another? Obviously, the question will usually be nonsensical, given the complexity of real-life faces.

To be sure, the above concept, while extremely useful, is probably too specific and tied to some original domain space to be of wide applicability. How about moving two forms into some sort of standard position before comparison? That might permit the use of the approach just discussed, for example. That strategy has been very common in artificial intelligence pattern recognition situations. It has been derided sometimes by theorists as to particularistic. However, Palmer (e.g., 1989, 1992; Palmer, Simone, & Kube, 1988) argues that objects possess intrinsic reference frames which allows shape comparisons to occur regardless of differences in orientation, size, position, etc. According to this reference frame hypothesis, objects can be brought into a canonical position and orientation so they can be appropriately matched. People can be pretty good at mental rotation (Shepard & Cooper, 1982) and there is some evidence that humans mentally mimic Newtonian mechanics or kinematics in their mental physics (e.g., Carlton & Shepard, 1990a, 1990b). So a rapid mental rigid transformation that brings two forms into as close an approximation as possible is not beyond the realm of possibility. It need not be a fixed position, but could depend on the two figures. For instance, a dynamic program operating on a hill-climbing, or gradient descent principle based on the disparity or distance measure, along with a class of rigid transformations (we are assuming rigid objects here), could make this procedure function effectively. Then, the final "minimum" disparity could be taken as the dissimilarity.

Obviously, we must mention the importance of the Gestalt principles of invariance and symmetry (e.g., Koffka, 1935) in all areas of perception and cognition. Recently, investigators have begun to employ these and more contemporary analogs in structural notions of perception and perceptual cognition (e.g., see Leyton, 1986a, 1986b; Imai, 1992; Buffart & Leeuwenberg, 1983; Chen & Chen, 1982, 1987). These may be fruitful in the enterprise initiated here, but space and time (and limited capacity on the part of the authors!) preclude further discussion herein.

Another more abstract possibility, at least for well known objects, is that there exists a memory representation that is an equivalence class of all possible "real" transformations on an object, that serves in comparison operations. This hypothesis is surely wrong in any universal sense, as indicated by difficulties in recognition of, say, faces when they are turned upside down (e.g., Bruce, 1988; Valentine, 1988).

Of course, if the comparison is based on easily measured features, or on more abstract principles (e.g., "give a rating that indicates the similarity in the manual use of the two objects"), then there would be an intermediary mapping that measures the appropriate quality or quantity, but does not necessitate this "physical" standardization strategy.

In the present context, it should be even more obvious than with finite dimensional stimuli, that preservation of specific geometric or topological aspects is arbitrary or at least peculiar to particular sets of forms as stimulus patterns and the requirements on the observer's processing systems. Thus, in some tasks the curvature (there are several types) of a form may be highly important and form a set of dimensions according to location, whereas in other tasks the number of holes or other qualitative aspects might be critical. Nevertheless, once we specify the information required for a task, then we can ask about dimensional and topological structure relevant to that task, and the preservation of such structure from external to internal representation. In this vein, as noted above, we consider same-different face discrimination to be virtually infinite dimensional since any two of the millions of faces present on earth can easily be distinguished. However, long-term memory of faces is quite another matter and the dimensionality of the underlying memorial space could be much reduced. Analogously, Todd and Reichel (1989) have questioned whether metric information is preserved in certain shape perception tasks, or only ordinal information. Such investigations are of great importance and we need many more in that spirit, optimally paired up with sophisticated theoretical tools.

To be sure, just as in the finite case, even if there has been preservation of external to internal metric (or premetric) structure, the observer's comparison model may well not preserve that information any further. Also, even if the observers are using a consistent dimensional strategy on psychological representations, there is no current multidimensional procedure for dealing with an infinite number of dimensions. We are somewhat dismayed that the ingenious results on the psychological scaling of Riemannian manifolds (Lindman & Caelli, 1978), apparently have not been followed up or much applied. However, these strategies too are meant for finite dimensional manifolds. As we have seen, a space of finite manifolds can be itself infinite dimensional.

We close this section by making a few remarks about the early dynamics involved in the percept of a manifold type of form. Knowledge regarding nonlinear dynamics in finite dimensional spaces is still growing by leaps and bounds, and is far from complete. That involving infinite dimensional spaces is much more inchoate. Nevertheless some things are known. For instance, the notions of asymptotic stability of a point (the trajectory approaches this attractor point as a limit across time) carry over nicely as do a number of other concepts. Hence, we can think of a percept of a form existing in an infinite dimensional space as approaching a stable percept after being presented.

Case IIB. Manifolds In Motion. This section will be very brief, but we do want to initiate discussion of situations where forms are in motion, for instance, rigid motion. Consider the example of a ballet dancer perceived over some time interval. From the great body of work stimulated by Johansson (1950, 1973), it seems very likely that even in situations where the facial and body structure precludes identification of the individual when motionless, that an expert choreographer can identify many performers simply from their motion. How can we capture this kind of perception in our theoretical structures?

Basically, the idea is that the trajectory itself of the dancer in motion and/or special features of that trajectory becomes the form. The dancer can, for now, probably be adequately described as a coupled set of individual parts moving in rigid motion. Eventually, describing more plastic motions such as a change of expression, require infinite dimensional systems, as for example, partial differential equations (see Beltrami, 1987; Zachmanoglou & Thoe, 1976). Again, differential geometry and its handmaid, tensor analysis, should aid greatly in this endeavor. Lappin and colleagues (e.g., Lappin, 1990; Lappin & Wason, 1991), W. Hoffman (1966) and D. Hoffman and Bennett (1985, 1986) have made progress on related themes, and the work of Caelli, Hoffman, and Lindman (1978a, 1978b) could also be helpful in this regard.

The similarity of two motion forms could be determined through the same kind of metric on functions, this time on shapes moving through space, that were illustrated above. Alternatively, as suggested by Lappin (1990), such features as the fundamental metric tensor (which characterizes the curvature of the space and permits the calculation of paths) may be abstracted by observers (e.g., Lappin & Wason, 1991; Lappin & Love, 1992).

A particularly exciting line of research, closely related to the present topic, goes under the name of "event perception." The literature is already growing vast but some starting references are Ullman (1979), Warren (1977), Shaw (1988), Rosenbaum (1975), Lee (1980), Cutting, Proffitt, & Kozlowski (1978), Algom & Cohen-Raz (1987), and Bingham (1987a, 1987b). In the more formal treatments, mathematical dynamic expressions enter explicitly into the formulation and predictions (e.g., see Bingham, 1991a, 1991b).

Case III. Extraction of Features

Now let us begin with a space of some variety, perhaps describable as a manifold, as many visual objects will be. It seems likely when first meeting a more or less novel form, that the entire form is allowed access to the repositories of visual long term memory, including outlying recesses. However, we may view this neurally and psychologically huge expanse as endowed with a distribution of salience across it. Only the most salient or "primed" locations will respond sufficiently to enter consciousness or to implement other associations. Of course, when a form belongs to a well established category, for instance, faces, then mechanisms often will go into play that focus on individual features. Furthermore, in learning to discriminate among a set of patterns, or identify them individually, it behooves a limited capacity system such as visual perception to reduce the processing load by allotting capacity only to a certain region or subset of the global information (e.g., Eriksen & Murphy, 1987; Bundesen, 1990; van der Heijden, 1975; Sperling & Dosher, 1986; Shiffrin, 1988; Townsend, 1981; LaBerge & Brown, 1989; Townsend, 1990a).

What is going on in terms of the present topological and geometric issues in this ubiquitous phenomenon? Interestingly, the dimensionality may not change. Consider the allocation of attention to a spatial subregion of the overall figure. If the form is three dimensional, then in many cases the subregion, wherein the sought-for feature may lie, will also be three dimensional. Thus, even though the focusing on a smaller region will typically require less capacity, the literal dimensionality of the space is equal to that of the entire form. In fact, one may envision cases where the dimensionality of the subspace, perceptually speaking, could be larger than that of the original perceptual space. For instance, suppose the perceptual concomitant of a colored form were composed only of the geometry of the form, without analysis of the color, perhaps due to limited capacity. Then the allocation of attention to subparts of the form could include color and thus augment the dimensionality of the form.

In other auspicious cases, the subspace will be of lower dimensionality. Thus, consider a situation where the observer must evaluate the curvature of a (not necessarily straight) line that lies on a two dimensional surface. In that event, the dimensionality of the line feature is definitely less than the dimensionality of the original surface. The human (and nonhuman, for that matter) visual system seems to be excellent at abstracting information of this kind from real or schematic faces—the well known efficacy even of poorly drawn cartoons being a prime example.

What is the topology of a subspace? Strictly speaking, in topology, a subspace of points inherits the so-called "subspace topology" of the original space. The subspace topology is formed composing the open sets of the new topology from intersections of the old open sets of the original topology with the subset of points under consideration. This accomplished, one may ask all the questions associated with any topology.

It is important to point out, however, just as the topology of the entire form may not correspond to that usually employed in our externalized model of reality, so the subset of points of the original form may not (in fact, usually will not) necessarily be a true topological subspace of the original form. Recall that the perceptual topology is derived from the perceptual task at hand, including the dimensions needed for comparison or other operation. This is perhaps most obvious in a relatively artificial situation, where the global contours are made up of smaller objects, sometimes with a completely distinct "meaning" (e.g., Navon, 1977; Kinchla, 1974; Uttal, 1988).

Even when the task involves discrimination among a set of forms, the topology may differ depending on whether global information is employed from each form, or whether features are extracted and used to make the discrimination. Thus, a global comparison might utilize the overall physiognomy of a head or face whereas, a featural comparison might evaluate the size and curvature of the lips. The topology and geometry would typically differ greatly in the two cases.

What about metric or qualitatively premetric information? Much the same points of discussion ensue as above. The mappings that carry pairs of features into similarity or dissimilarity ratings, and sets of such pairs, will typically be based on individualistic "distance" measurements, according to the specific features. Thus, a comparison of eyes of people will often include a measurement of hue, whereas that of ears typically would not.

After learning has occurred, or under instruction, an observer may decide to allocate attention to a selected set of features. If this set is sufficiently small, compatible, and specified with respect to relative spatial position, then the processing may be in parallel, otherwise, serial processing may have to be implemented. More detailed recent models of attention will be helpful in bringing together this information processing aspect of perception with the topological and geometric aspects (e.g., LaBerge & Brown, 1989; Bundesen, 1990). However, we need much more work in this area, especially as concerns the synthesis of information processing concepts with the topological. For instance, Massaro's (in press) review of information processing research over the past decade reveals a growing, and long awaited, concern with the state space and type of coding, in human information processing systems. However, there is little evidence of the kind of synthesis to which we refer here. Exceptions are the research tracks followed by Ashby and colleagues (e.g., Ashby, 1989; Ashby & Gott, 1988; Ashby & Perrin, 1988) and Nosofsky and associates (e.g., Nosofsky, 1985, 1986, 1987; Nosofsky, Kruschke, & McKinley, 1992) where elementary dimensions of the percepts compose a critical part of the perceptual operations and decisions. Biederman and colleagues' theory of geons is close in spirit to the present topic and the two perspectives probably have much to offer one another (e.g., Biederman 1987; Hummel & Biederman, 1992). The line of research on dimensional separability vs integrality and independence vs dependence (e.g., Garner, 1974; Shepard, 1987; Ashby & Townsend, 1986; Kadlec & Townsend, 1992a, 1992b) can also play an important role in this challenging domain.

CONNECTIONIST MODELS AND THEIR GEOMETRIC IMPLICATIONS

There are more connectionist models out there today than one can reasonably shake a stick at, but most of them rely at least in part on applied mathematical strategies that have proved of use in engineering and applied physics, particularly in pattern recognition fields. (To be sure, cognitive science theorists have made important contributions to the ongoing theoretical developments as well.) For instance, an early linear transformation of the input is common (e.g., Kohonen, 1989; Kruschke, 1991). What if we assume that the physical space, U, is a finite dimensional Euclidean vector space and that the early transform of U to X is indeed linear, $X = AU$, where A is a matrix?

First off, we have noted earlier that even this simple-minded transformation will, in general, alter the order of the pattern distances. Now suppose that the observer performs some comparison process which we can express as $y = C(x_1, x_2)$ for two perceptual pattern vectors x_1, x_2 and where y is a number. (More generally, y could simply be a member of a simply ordered set.) Under what circumstances will y be a distance, or at least monotonic with distance? A favorite "chestnut" in the literature on pattern recognition is to let C be the dot product (known in psychology, somewhat inappropriately, as the cross-correlation; cf. Papoulis, 1965) of x_1 and x_2. In identification operations, a second stage linear mapping to assess the strength of each candidate response, turns out to be optimal if the associated dot product has certain properties (e.g., Nilsson, 1977; Fukanaga, 1990). The fact mentioned in an earlier section, that under certain circumstances, it yields the optimal matched filter, as well as the minimum distance classifier is one example. Hence, if such situations are transferred to similarity matching, then the correlation operation can be viewed as monotonic with distance.

Such questions can be asked in the domain of general comparison functions, C. Such functions bear a close resemblance to discriminant functions, which are enacted on incoming patterns in identification paradigms, in order to classify them (e.g., see Kohonen, 1989; Townsend & Landon, 1983; Ashby, 1992, ch. 1). And, in fact,.the questions raised in the present context, that of similarity matching deserve to be raised in other paradigms such as identification and categorization. In any event, let us consider another alternative, that where C is a so-called quadratic form. That is, $C(x_1, x_2) = x_1 \mathrm{B}\, x_2$ where B is a square matrix. Interestingly, if B is positive definite [which means that $C(x_1, x_2)$ as defined above is zero when $x_1 = x_2$ if and only if $x_1 = 0$, the zero vector], then we can used it to define a norm. [A norm is a measure of the magnitude of the vectors, represented by the positive numbers such that it is zero if and only if the original vector is zero, if multiplication of the vector by a number is the same as multiplying the norm or the vector by the absolute value of that number, and if $norm(x_1+x_2) < norm(x_1) + norm(x_2)$, that is, the norm is subadditive.] The norm can then be employed to define a metric simply by taking the norm of the difference between two vectors. It can be shown that this operation meets all the requirements of a metric, including the critical triangle inequality. However, it should be noted that the quadratic form imposed on two vectors to, say, measure similarity, will not be symmetric in general unless the matrix B is symmetric.

Quadratic forms implemented on two comparison vectors, or on their difference vector to compose a distance, are easily realized in neural networks, and hence form a viable modeling possibility. It is intriguing to the present writers, that these fairly common types of functions are so close to being metric computations. If the sensory apparatus and brain actually perform something like these operations, we may have one good reason why multidimensional scaling programs work as well as they do, even without taking into account a process model of whence the geometry arises.

Still the basic conundrum is not quite resolved since as shown above, it is almost trivially easy to disrupt even the order of distances (or premetric concomitants of distance). The worst case scenario would be that our *MDS* routines are acting as sophisticated Rorschachs, allowing us to "unveil" the geometry we seek rather than the psychologically "true" space. The best case would probably be that even though neural transformations may be distorting the distance orders (not to mention the actual distances), the perturbation is sufficiently minor that *MDS* software is able to rectify the damage.

It appears that more work on linear and nonlinear operations employed in topical connectionist and other process models, with regard to their effects on physical and perceptual geometries, could be quite valuable.

DIFFERENT TYPES OF SEPARABILITIES

During the earlier discussion of the example of pitch and loudness perception we alluded to the issue of separability versus integrality that is of interest in modeling perceived similarity. Such questions have a deep history in the philosophy of thought, culminating in the nineteenth century in the opposing dialectic camps of German phenomenology vs British Empiricism. The former emphasizing the global uniqueness of thought products (and incidentally their innateness) while the latter endorsed the presumed construction of thoughts and concepts from atomistic pieces (and incidentally, the criticality of learning and impact of the environment). Of course, these philosophical roots gave rise on the one hand to Gestalt Psychology and on the other to Structuralism (early) and Behaviorism (later). The modern approach embodied by such investigators as Roger Shepard and Wendell Garner, has implicitly been something of a synthesis of the Gestalt and Structuralist position, since by definition, a pair of stimulus dimensions might be either integral (i.e., Gestalt-like) or separable (i.e., like two independent atoms). Views on separability in the context of multidimensional scaling during the 1960s and 1970s often centered around the type of metric that best described similarity relationships of the stimuli under study (Shepard, 1964; Garner, 1974). Separable dimensions were defined, somewhat informally, as those that remained distinct and analyzable when processed and selective attention could be given to them. Also, when a multidimensional scaling solution was derived for proximity data, the city-block metric fitting best was an indicant that the stimuli were composed of separable dimensions.

In contrast, integral dimensions were deemed to be unitary, unanalyzable wholes which could not benefit from selective attention and the Euclidean metric was associated with this type of dimension. We have already noted the questionable nature of associating a particular metric with certain types of dimensions. Also, problems with this characterization of dimensional interactions are well laid out in Ashby and Townsend's (1986) introduction to general recognition theory. Part of the trouble with the traditional view is that perceptual effects were not teased apart from decisional effects. Stimulus dimensions may or may not interact perceptually and perceptual dimensions may or may not be combined when arriving at response decisions.

In addition, perceptual dimensional interaction may take the form of a dependency of dimensions within a stimulus or an integrality of dimensions across a stimulus set, these actually forming distinct concepts. The formulation of general recognition theory (Ashby & Townsend, 1986) helped to exhibit this distinction. If two stimulus components or channels were independent within a stimulus set and within observation trials the concept was termed "perceptual independence" and is due to probabilistic independence or the absence of a (possibly nonlinear) correlation in the statistical sense. An invariance of a components characteristics across stimuli was dubbed "perceptual separability." These concepts had their inception in our earlier treatment of perceptual effects of features in letters (e.g., Townsend & Ashby, 1982). A theoretical structure such as general recognition theory also helps to relate several different notions of separability and integrality in a common framework.

In the absence of such a framework, different operational definitions of separability and independence can lead to seeming contradictory empirical findings when the various concepts are confused. For example, Zagorski (1978) outlines such contradictory separability results in the context of auditory pitch perception. Scaling studies (Carvellas & Schneider, 1972) and filtering tasks (Garner, 1974) indicate that the dimensions of pitch and loudness were integral and yet other psychophysical evidence indicated that they need not interact in certain tasks (e.g., Zagorski, 1978). This apparent paradox was dispelled somewhat when he observed that perceptually the dimensions showed evidence of independence yet decisionally they would often be combined prior to the subject making a response. With only the type of metric available to them, it is unclear how *MDS* based models can capture this distinction between the source of interaction unless there is a concomitant process model proposed (as in general recognition theory or Nosofsky's generalized context model, Nosofsky, 1986). In any event, when examining the issue of perceptual separability, in general, one can appreciate that there may be several levels of dimensional separability throughout the cognitive processing chain.

At the very beginning, a failure of separability of the physical dimensions may occur directly in the transformation from the physical space to the psychological space. This was the case in the pitch-loudness example in that the function $\rho: (x, y) \rightarrow [f(x, y), g(y)]$. This is perhaps the most interesting and yet most difficult type of dimensional interaction to identify. However, if present, it would surely be evidenced in more standard means of analysis. For example, when one changes the frequency of a tone, both loudness and pitch would change. Hence, loudness and pitch could be considered to be linked in some way. This would presumably be detected by a *MDS* analysis. For example, using the Euclidean metric, stimuli generated by orthogonal combinations of frequency and intensity would never line up in a rectangular configuration in the derived solution.

Another way to explore separability is to examine the perceptual space post transformation. One can ask, within this space, can all points in the cartesian product of the two (now psychological) dimensions be realized? In the case of loudness and pitch perception, this is almost true. Every loudness can, in principle, be experienced with every pitch with *one exception*: every loudness has to have *some* positive pitch, and vice versa.

Some have used this kind of existence dependence as a defining property of integral dimensions. However, this may be too severe. When both dimensions have positive value, one could maintain a constant loudness and vary pitch, but both frequency and intensity will have to be changed. For example, in the midrange, if pitch is to be increased, frequency will be adjusted upwards while intensity will be need to be decreased in order to maintain a constant loudness. Nonetheless, almost all points in the loudness x pitch space are possible perceptions. This is not so in the case of color perception. The full cartesian product of hue, saturation, and brightness is the volume of a cylinder. Much less than this full space is perceptually realizable. This can be seen in Figure 3 which contains a simplified rendering of the traditional color solid as a subspace of a cylinder (Munsell, 1915; Newhall, Nickerson, & Judd, 1943).

Other often used stimulus dimensions are the size of a (semi)circle and the angle of incline of a radial line. These dimensions have often been shown to be separable by standard techniques (Shepard, 1964; Garner & Felfoldy, 1970; Hyman & Well, 1967; but see Ashby & Maddox, 1990; Nosofsky, 1985). They also are able to enjoy the full cartesian product of size x angle possibilities. Perhaps it is this type of separability that is at work in the standard Garner filtering task. In this task, separability is implicated if varying irrelevant dimensions (irrelevant as defined by task demands) does not interfere with processing the relevant dimension(s). From one point of view, psychological dimensions are now orthogonal, in the sense that the full cartesian product is available, and, hence, we should be able to pick points (e.g., a line of varying pitch orthogonal to a line of constant loudness, say) such that no interference occurs.

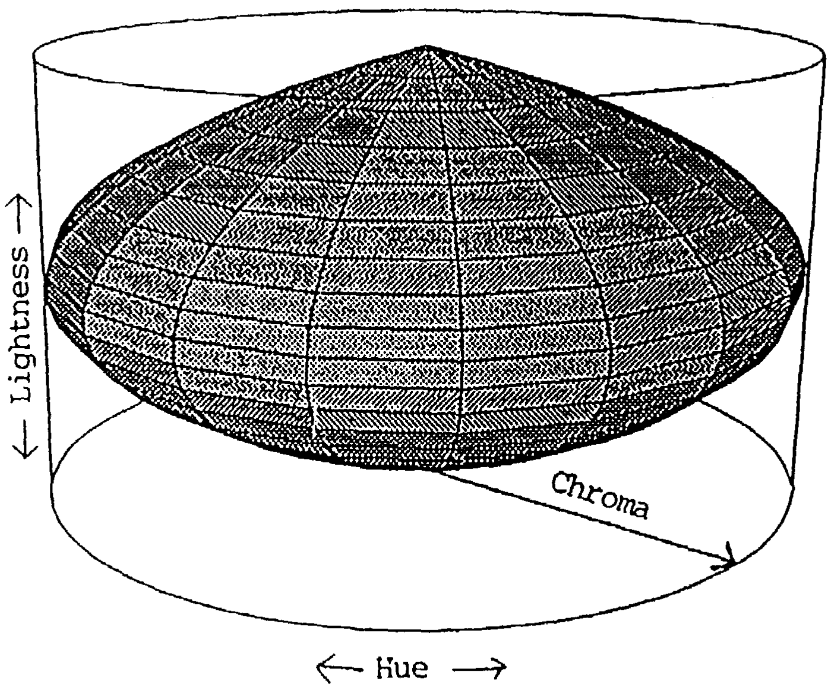

Figure 3. A simplified Munsell color solid sitting inside a cylinder. The full cartesian product of hue, saturation (i.e., chroma), and lightness corresponds to the volume of the cylinder. The color space is a proper subset of this cartesian product and, hence, leads to a violation of a kind of separability.

In their early work on the foundations of multidimensional scaling, Krantz, Tversky, and colleagues (Beals, et. al., 1968; Tversky & Krantz, 1970; Krantz & Tversky, 1975) argued that a certain kind of separability was necessary to even define psychological dimensions. Dimensions, in order to be considered valid psychological dimensions, had to contribute additively to the overall distance between stimuli. This interdimensional additivity property is analogous to the requirement of conjoint measurement in that variables need to have additive effects on some measure in order to be considered "independent" in their contributions to that measure. Their actual mathematical results were posed in terms of transformations of physical scales. The discussion at the end of the Tversky and Krantz (1970) paper, which advised the operational defining of psychological dimensions in terms of criteria such as interdimensional additivity, taken in conjunction with their tangible mathematics has an implication. It implies that their results on scaling, actually work in a direct fashion only if the physical scales map in a "separable" fashion, into the psychological dimensions. Otherwise, the investigator would first have to ascertain the laws of interaction (i.e., the maps carrying the physical dimensions into the psychological dimensions), and then show that these new dimensions exhibited interdimensional additivity.

With regard to the metric criterion, it is interesting to note that all Minkowski metrics satisfy interdimensional additivity. That is, both the city-block and the Euclidean (among others), from the conjoint measurement point of view, are consistent with a type of separability. In any case, the metric argument itself, based on the peculiarity of the city block distance, might be strengthened by embedding it within a process model. For instance, suppose that early processing of stimulus dimensions satisfied some of the early requirements discussed above, for separability. Then suppose that evaluation of similarity took place within the observer by way of a serial operation which computed the absolute value of differences on the separate dimensions and then combined this information additively. This model gives a psychologically plausible sense to the fashion in which a metric could be associated with "separability." On the other hand, it is of course true, that cognitive systems could act much like that scenario, for instance operating serially on feature sets, without producing a metric result.

FURTHER THOUGHTS ON FOUNDATIONAL MEASUREMENT VIS-A-VIS PROCESS MODELS

An Achilles heel of foundational measurement is widely believed to attach to its as yet undeveloped statistical underpinning (e.g., see Cliff, 1992; Falmagne, 1992; Marley, 1992). Due to this lacuna, aside from the mathematical beauty of some of the literature, it is safe to say that many of the implications have been within the arena of "hard" sciences, especially Newtonian physics. As far as the psychology is concerned, the primary drive has perhaps been "philosophy of science" in nature, although there have been important inroads into decision theory (e.g., Fishburn, 1982; Krantz, et. al., 1971; Suppes, et. al., 1989). From this point of view, the results that relate to psychological geometry could turn out to be of most direct impact to the behavioral sciences. Although probabilistic aspects will likely play an important role in the latter, their scarcity in the present context does not seem to be so troublesome.

Nevertheless, there are a number of open problems that remain to be successfully attacked, several mentioned in the foregoing discussion. The linkage of foundational measurement geometry with process models and multidimensional scaling analysis is one that has been harped on throughout this account.

First, the skeptical process modeler or experimentalist might ask whether, given that the qualitative structures proposed by the measurement investigators are sometimes so close to being a metric space to start with, and so easily disrupted by possible psychological transformations, every-day researchers should undergo the arduous training required to actually work with (as opposed to "culturally experiencing") the theory. The authors of this chapter plump down on the affirmative side, but the question merits further debate. Unfortunately, as with a number of technical developments in the behavioral sciences, many psychologists appear to possess little to no knowledge of this branch of research, even in regions where results can affect their own conclusions. An example is the regular abuse of psychological scales in empirical and psychometrics research (e.g., see Townsend & Ashby, 1984; Luce, et. al., 1990; Roberts, 1979).

There is another potential research area that we have not yet touched on. This topic concerns the development of process models that lead in a natural way to qualitative measurement structures of the sort studied by these investigators. If all the process models are always expressed in terms of nice traditional real-valued functions in well-behaved (usually Euclidean) metric spaces, how shall we ever bridge the gap? Just to state that psychological structures might be weaker qualitative entities, without knowing how they might arise, leaves a serious vacuum. Does a weakening of structure occur due to stochastic noise, or are we missing some fundamental link, perhaps a new way to represent massive neural networks or process models in general, that allows a proper synthesis and mutual benefaction?

On the other hand, some researchers have stated that as long as, say *MDS*, works in the sense of presumably revealing the underlying psychological space, then they are unconcerned with the kinds of questions brought up in this paper. We certainly agree that methods can work in a pragmatic sense, long before their underpinnings are secure. A case in point is the Heaviside function (e.g., see Lighthill, 1958) which physicists used for fifty years before it was endowed by mathematicians with a rigorous setting.

However, in the present situation, we feel that there are purely empirical reasons to pursue such theoretical goals. First, the revealed psychological space must have "traveled" from whatever process last contributed to its present characteristics, without almost any perturbation, as shown in earlier arguments above. Furthermore, the nature of such transformation, from that particular representation, to the output, must surely be of interest to psychology. The same should be said of the transformations from the physical world, through the sensorium and pertinent cognitive processes. Thus, we would propose that such investigations hold considerable value from a scientific, and simply mathematical, perspective.

Finally, there is a small set of literature that seems quite at home within the foundational setting, and in some ways, seems like a very natural way to approach the goal of providing an underpinning for *MDS*. It appears not to have received much notice to date. We will refer to it as the "distance set approach" because that rubric is common in the attending papers. Basically, the investigator considers a set of positive (possibly including 0) numbers and asks whether it can realized as a set of distances in a certain space or class of spaces. Clearly, that is, to put it colloquially, "part and parcel" of the *MDS* program. Although some of the theorems are naturally somewhat esoteric, some of the others are rather striking in their implications.

For example, Kelly (1951) demonstrates the existence of $n+k$ (k even and positive including 0) numbers that cannot be distance sets for any subset of an n-dimensional Euclidean space! But now the conundrum of monotonicity arises yet again. That is, if we assume that internal psychological information may have been perturbed so that empirical data are monotonic transformations of the psychological distances, then it must be okay to re-transform them to regain the psychological space. But then the investigator is in danger of transforming data that itself cannot be put into any, say, Euclidean space into virtually anything, and in fact something that cuddles quite nicely into an n-dimensional Euclidean space. On the other hand, there exist even infinite sequences of non-negative numbers that can be embedded in many spaces. The upshot is that there seems to be little that is ruled out when one permits oneself to perform arbitrary monotonic transformations on data. One of the hallmark papers in this literature is Kelly & Nordhaus (1951), who also list a number of relevant papers. Some of the names are well-known in mathematical circles, such as Erdös, Sierpinski, Coxeter, and Picard.

Thus, again we have to face the importance of considering what transformations are actually occurring within psychological processes. Such knowledge, and the building of such constraints into our process models of psychological geometry, might be conjoined with the mathematical results of the distance set approach, to not only produce effective *MDS* transformations, but to place restrictions on what those transformations might be. The fact that the *MDS* functions that are actually employed (in contrast to any arbitrary monotonic transformation) turn out to be a restricted set (e.g., see Young, 1975, and Bentler & Weeks, 1978), might be highly useful when analyzed in conjunction with probable classes of psychological transformations.

The paragraph above illustrates the associating specific metric assumptions to particular *MDS* procedures and these together to process model concepts. An alternative tactic, well demonstrated by Holman (1978), is to employ only ordinal betweeness relationships in estimating positions of objects within an n-dimensional coordinate system (but sans metric or even additional premetric assumptions). From one point of view, Holman's (1978) paper realizes the goal of a purely "nonmetric" scaling methodology.

CONCLUSION

In the preceding sections, we have ranged over some of the important mathematical concepts underpinning physical and psychological similarity and distance. These concepts were discussed in the context of psychological process theories, particularly with regard to preservation of characteristics that would allow the embedding of similarity or dissimilarity ratings (or other like dependent variables) into metric spaces. It was emphasized that such concepts as dimension and feature, not to mention metric, are meaningful only when the psychological functions of a specific task, and therefore also the implied stimulus variation, are taken into account.

Once the hypothesized psychological elements of a task are made manifest, plausible psychological models can be implemented and their topological and geometric implications, in addition to their usual predictions, studied. Within this milieu, some aspects of similarity as relating to finite and infinite dimensional spaces, to objects in motion and to feature extraction were reviewed. Yet another benefit of such an enterprise might be the resolution of the paradox of most information process models: the form of the information and what happens to it, are usually absent from the models. Taking in account the transformation of spatial information (in the broad sense) of stimuli, and throughout the psychological system, could help to eliminate the paradox. In conclusion, it seems clear that the time has arrived for more intensive investigation of the triad, foundational measurement theory, psychological process theory, and numerical multidimensional scaling methodology, taken in concert.

Finally, a few remarks are in order regarding the overall goals of the present tome. The fact that three of the most rigorous approaches to a specific perceptual topic have shown so little evidence of convergence, is not especially encouraging with respect to the much more ambitious aim of a unified, and perhaps reductionistic, theory of perception in general. What is the outlook for such a general theory? A high degree of speculation is in order here, and our predilection for formal theory will be apparent.

We foretell (soothsay might be more apt) that in the indefinite future, a loosely knit, hierarchy of theoretical structures on human cognition will come into being after many, many decades of challenging empirical and theoretical effort. This macro-structure will be defined by a complex interplay of dynamically defined subsystems. Particular systems, such as "memory" or "form perception" will be thought to engage vast, though somewhat separable neural processes. Their theoretical definition will depend on the one hand, on empirical task demands, and on the other, with hypothesized (and partially verified) dynamical interactions among subsets of functional subsystems [which in turn, may depend on wide-spread and in some cases distributed (in the connectionistic sense) neural elements]. That is, one hallmark of the grand theory of perception, will be that specific subprocesses of perception will rarely be strictly definable by single, unique and self-sufficient modules (although researchers may have recourse to that wrong assumption for certain useful approximative purposes), much less by ineluctable physiological structures.

At a single level of analysis (a "horizontal" section of the hierarchical macro-theory), theory and spin-off models will be reasonably tightly defined and connected with one-another. For instance, at a moderate process level, let's say, the regions of interface between early visual processing of symbolic stimuli and the systems of short-term memory will be rigorously specified. However, the (vertical) ligaments connecting large-scale phenomena with more fine grain, for instance, physiological, operations will exist in principle, but will be fuzzy and only invoked in detail in the presence of great need, and under considerable duress for the scientist. The concept of similarity will exist in various modes, related to the special subsystems and the cognitive task under consideration. The disparate similarity incarnations will also possess varying degrees of quantitative and metric properties. The implications of the overall, as well as that pertaining to similarity, scenario for truly scientific applied spheres of psychology, for instance, human factors and applied decision research as well as clinical psychology, are quite fascinating, but for now, we may ponder at the local pub.

APPENDIX

On the Mathematical Definition of Dimension

Here we present a more rigorous discussion of dimension using the "covering dimension" definition developed originally by Lebesgue (see Munkres, 1976). We need to introduce several preliminary notions. The following definitions are taken after Munkres (1976).

Definition: Given a collection Λ of subsets of a space X, another collection Ω of subsets of X is a **refinement** of Λ if each element ω of Ω is contained in (i.e., is a subset of) at least one element of Λ.

One can think of the refining process as crushing the original sets into "smaller" ones, like stones into pebbles; hence the term refinement. The pebbles can be thought of as being contained in some larger stone.

Definition: A collection Λ of subsets of a space X has **order** $m+1$ if there exists a point of X that lies in $m+1$ elements of Λ but no point of X lies in more than $m+1$ elements of Λ.

Captured in this definition is an intuitive notion of the overlap of sets. Suppose the collection in question has order $m+1$, then the maximum number of sets allowed to "overlap" in one place is $m+1$; but, there has to be at least this many ($m+1$) sets overlapping in some place. Now we are ready to state the formal definition of dimension due to Lebesgue.

Definition: A space X is **finite-dimensional** if there is some integer m such that for every possible collection Λ of *open* subsets of X whose union contains X (also known as an open covering of X), there is another collection Ω of open subsets of X whose union contains X, that is a refinement of Λ and has (i.e., Ω has) order at most $m+1$. The **topological dimension** of X is defined to be the smallest value of m for which this statement holds.

An example will help to illustrate this rather abstract definition. We know that the plane, $\Re^2$, is a two dimensional space. Now, let us use the definition to corroborate our knowledge. In Figure 4a, we have covered the plane by open circles which overlap each other in various places. The order of this collection of circles is four since in the shaded region four sets overlap. This tells us the plane is finite dimensional because four is a finite integer. What is the least such integer, m, here? Consider shrinking (or crushing) the circles into smaller overlapping circles keeping in mind we need to cover all the points in the plane. In Figure 4b we see this refining process results in the order now being three. That is, the maximum number of overlapping sets anywhere is three but there has to be some place in which three sets overlap or else some points in the plane would not be covered. This can be seen in Figure 4c. Here, the open sets have been

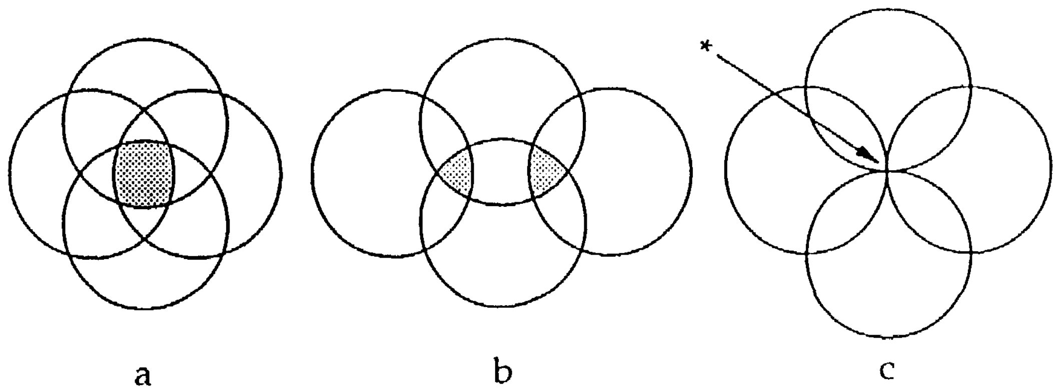

Figure 4. (a) A local view of one possible covering of the plane. The area of maximal overlap is the shaded region which corresponds to the overlap of four sets. (b) A refinement of the previous covering. Now, the maximal overlap (shaded regions) corresponds to only three sets. (c) A further refinement that cannot be a covering because the point, *, is not in any of the open sets. We see from this series of refinements that the dimension of the plane is two.

refined so much that only their boundaries touch. But, note that the boundary of a given disk is not in that set because the set is open. Thus, in this particular refinement, the point in the center is not in any set, so this is not a valid covering of the plane. To include that point, we need some overlap of at least three of the sets. Hence, we conclude that the plane has dimension two.

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DISCUSSION

Vicki Bruce and Mike Burton (*Department of Psychology, University of Stirling, Stirling, Scotland*): Townsend and Thomas have presented a detailed technical review of some of the basic issues that arise when trying to formulate an appropriate account of pattern similarity. In order to understand the nature of the psychological space lying at the front end of perception, we need to understand the mathematics of the possible spaces and mappings that could relate physical dimensions such as frequency and intensity, to psychological dimensions such as "loudness."

The issues raised are complex, and somewhat daunting for the jobbing experimental psychologist trying to understand object perception. More important than their complexity, however, is the fact that it seems to us to be extraordinarily difficult to apply these underlying approaches to similarity spaces to the kinds of percepts that many of us find interesting. In most of their essay, Townsend and Thomas consider the perception of stimuli which can readily be understood to be manufactured from some combination of values on a set of comprehensible physical dimensions, such as the frequency and amplitude of a pure sinusoidal tone. Given the clarity with which the physical space can be described, it is appropriate to ask how this space translates perceptually, and how the perceptual space may then be used in turn to mediate activities such as similarity judgment or identification.

From time to time in their article, however, the authors referred to a class of stimuli that seem to pose a quite different set of issues when it comes to uncovering the nature of psychological perceptual space and how this is used to influence behavior. Human faces are but one example of the more naturalistic kinds of forms that humans evolved to perceive and recognize. What messages can be gained from this chapter, and the literature on which it draws, for understanding the perceptual dimensions which we use in judging faces, or other complex everyday objects and sounds?

The issues reviewed in Townsend's and Thomas' chapter are timely, as notions of a psychological similarity space for faces have been increasingly dominant in recent years. Researchers have moved from charting the cognitive processes of facial identification (e.g., Bruce & Young, 1986; Young & Bruce, 1991) to exploring the perceptual "front end" on which the later cognitive activities are based. Research on facial distinctiveness (e.g., Valentine, 1991) and on caricature (Rhodes, Brennan, & Carey, 1987) has been explicit in suggesting that faces can be located in a "space" of facial variation, which underlies both perception and memory for faces. The idea is that the more extreme and isolated is the location of a face within this space, the easier it will be to identify. Moreover, the caricature hypothesis suggests, and is supported by evidence, that a face can be made artificially more distinctive by making it more deviant from the norm. To date, the geometrical approach to "face space" has been unashamedly Euclidean. For example, Valentine (1991) assumes a Euclidean space "for simplicity," awaiting evidence of a more appropriate geometrical model. Rhodes et al. (1987) develop a model of caricature in which a face can be made more distinctive by exaggerating its difference from the "average" face. The "difference" which is exaggerated is the location of each of a large set of x, y coordinates of key points on contrast contours, and the exaggeration is obtained by simple multiplication of these x, y distances by a caricaturing parameter, which can be positive or negative. Moreover, resulting caricatures are expressed as percentages according to the value of this parameter and ordered linearly as values of the independent variable (degree of caricature). We believe that a number of problems of interpretation of results arise from this approach and we would like to see investigators in this field take issues of geometry and scaling much more seriously. This is particularly because the results are so intriguing and provocative, for example Benson and Perrett's (1991) extension from line-drawn to photographic caricatures, where they appear to be able to produce caricatured images which look more like people than the original, veridical images!

Another area where related issues arise lies in methods based upon principal components analysis (*PCA*) of the variations in intensity (gray) levels from one face image to another. Some of these approaches have in recent years proved rather interesting (e.g., Turk & Pentland, 1991; see Bruce, Burton, & Craw, 1992, for an overview) since they seem to provide a potential model for face "space" from which the underlying "dimensions" will emerge simply from the statistical variation of different images of faces. However, the *PCA* methodology rests on the assumption that the data represent points in a linear space. As a result, all published work using *PCA* requires images to be carefully normalized, in terms of size and location of the face within the image and in terms of viewpoint. There is considerable scope for the development of more sophisticated models of space, including notions of a face manifold embedded within a higher-order space, to underpin this methodology (I. Craw, personal communication).

Can we make progress by more basic empirical work to uncover the nature of the similarity space for faces? The problem is that it is very difficult to uncover the possible form or dimensionality of such a space without making assumptions about the answer, as we do not properly understand the possible physical dimensions of variation which may be manipulated or revealed in our mappings. What are the physical properties of faces that correspond to the frequency or intensity of a pure tone? Can we make progress here by studying the effect of manipulations made upon face images to judgments made of them? At present, we think not. To date, there has been an assumption that manipulating the distance or size of "features" in the picture plane of the facial image represents a legitimate operation on the "stimulus," whose effect on perception can then be judged. However, an increase or decrease in the number of pixels separating the eyes in a picture of a face corresponds to nothing whatsoever in the physical space of faces, since faces are not flat patterns with *2D* distances separating their "features" but are actually bumpy surfaces. The pixels separating the eyes in a picture of a face convey, in their pattern of lightness and darkness, information about the protuberance and shape of the bridge of the nose. Altering their separation in the picture (i.e., x, y) plane may have the effect of making the nose bridge appear to recede or protrude more in the depth (i.e., the z plane). In recent years we have been arguing that a proper psychophysics of face perception must take account of the nature of the objects of face perception, and we have been trying to develop a methodology in which we examine perception as a function of changes to surface shape rather than picture layout (e.g., Bruce, Burton, Doyle, & Dench, 1989; Bruce, Burton, Hanna, et al., 1992; Bruce, Coombes, & Richards, 1993). However, this in itself poses problems since we must then develop the appropriate geometry to describe the face surface, and find ways of relating changes expressed in these geometrical terms to perceptual similarity, a difficult problem which we have only just begun to address in a highly simplistic way (Bruce, Coombes, & Richards, 1993).

Perhaps we can sidestep the issues by using multi-dimensional scaling of similarity judgments made to faces, to uncover the perceptual structures "directly" without manipulating the physical shapes of faces at all? Unfortunately many such studies have made use of schematic or highly simplified faces, artificially constructed and manipulated along artificial dimensions, thus raising the same issues addressed above (e.g., Sergent, 1984). Some investigators have used real faces, but have had to interpret the resulting similarity space in terms of proposed underlying dimensions (Shepherd, Davies, & Ellis, 1981). The results of such studies are highly dependent on the size and heterogeneity of the set of faces whose similarities are assessed. Although studies of real faces seem consistently to pull out dimensions related to overall age, face shape and hair length (for male faces), they have not yet yielded any finer scale information than that. This is presumably because resulting dimensions must be interpreted in terms of underlying face structure—and we have already seen that we do not properly understand that.

The complexity and (assumed) high dimensionality of faces makes the task of assessing similarities rather formidable for the observer. In this case, an alternative approach would be to try to understand the physical basis of a related variable, facial distinctiveness, which observers seem moderately comfortable with. In recent work (Bruce, Burton, & Dench, 1993) we have tried to relate psychological dimensions such as the measured memorability, or rated distinctiveness, of faces, to their objectively measured "eccentricity" in terms of their physical deviation from normal measurements. Reassuringly, there does seem to be a reasonable correlation between psychological distinctiveness and physical eccentricity, at least provided faces are both measured and judged without hair visible. However, although such studies avoid the need to manipulate faces, all the same problems that we discussed above are inherent in attempts to measure faces in terms of the spatial layout of features in the picture plane. In our study we tried to reduce these problems by including measurements of the surface protuberance as well as simple "2D" layout, but again only in the most simplistic way.

Are these problems peculiar to faces? (It has often proved convenient, when face perception proves tricky, to dismiss it as "special" and move on!). It is certainly the case that we must keep the different uses made of facial information clearly in mind when we consider what makes faces similar or dissimilar to one another. We may judge two different variations of the same face as very dissimilar one to another if the task is expression-based, but as very similar if the task is identity-based, for example. An ordinary-looking person can pull an extraordinary face. Few other materials are complicated by their multiple uses in social perception. However, faces are not the only class of objects within which we can make fine discriminations on the basis of extended experience, and all the problems we discussed above apply even if we restrict ourselves to the similarity space of full-face images of faces wearing neutral expressions. We recognize makes of car, breeds of dog, our own cars, our own dogs, as effortlessly as faces, given appropriate motivation and experience. For such complex stimuli the same questions arise of how we begin to study the basis of such perceptual judgments without making simplifying assumptions that beg the very questions we seek to answer.

Townsend and Thomas: Bruce and Burton raise a number of provocative issues in their insightful commentary, and provide several key references to face perception literature that relate to our chapter. Their emphasis seems to be on the spatial aspects of object perception. Bruce and Burton believe that faces and other naturalistic objects pose different sets of issues than simple psychophysical stimuli. We agree that they pose more complex issues but we are not convinced they differ in kind from those confronting the early psychophysicists. Basically, psychologists would seem to be interested in attributes of the stimuli that are employed by observers, how these attributes are transformed, including the Gestalt itself, and how these all interrelate to various psychological facets of behavior.

Naturalistic perception is more complex because observers have an extremely large number of "features" from which to choose in everyday situations. And, they very likely store and use holistic (and possibly hologramatic) Gestalts of frequently encountered objects like faces. Much of our own interest relates to high dimensional objects like faces, but we had to set the stage for discussion of such phenomena within the context of simpler objects. Bruce and Burton question the validity of "studying the effect of manipulations made upon face images to judgments made of them..." (p. 346). Part of the difficulty is that two dimensional variation may be having unknown or at least uncontrolled three dimensional effects, in many studies. However, this would not by itself appear to scuttle the general methodology itself so much as impugn ineffectual application. We have not yet acquired a couple of the intriguing "in press" references alluded to by the commentators but they seem to be quite compatible with the general milieu and format provided in our chapter. This may be the case with the "... notions of a face manifold embedded with a higher-order space..." (p. 345), and with "... we examine perception as a function of changes to surface shape rather than picture layout" (p. 346) (e.g., Bruce et al., 1989, 1992).

In our view, certain tasks likely elicit the full infinite (i.e., so large as to effectively be modeled by an infinite space) dimensionality inherent in objects like faces. Such tasks as a familiarity judgment regarding a face from a random crowd probably employ a similarity matching process that integrates over all of the available facial surface as well as hair color and the like. Another candidate for high dimensionality would be a same-different task. Although on any single trial, a certain region of the face or particular feature may suffice to distinguish two different faces, it seems probable that the entire facial surfaces and full set of characteristics may be compared in an unlimited capacity parallel fashion. On the other hand, a teacher meeting a new classroom full of students for the first time may utilize a few features to discriminate similar faces, in a limited capacity parallel or even serial fashion, until learning permits more efficient strategies. Of course, as noted in our paper, most "features" in natural visual stimuli will in reality be of higher dimension. Thus, the visible nose is itself a two dimensional manifold, so that the space of noses is an infinite dimensional manifold.

Bruce and Burton and their colleagues have shown us it is possible to make valuable strides in regions formerly eschewed as "too complex," such as face perception. We believe that the concepts discussed in our chapter may aid in approaching some of the goals resident in such domains.

David LaBerge (*Department of Cognitive Sciences, University of California, Irvine, CA*): Townsend and Thomas use the notion of object similarity to take the reader on a guided tour of a wide variety of mathematical structures that have been used in modeling object perception. The major areas of research, heretofore operating relatively independently, have provided formal structures for the psychological scaling of similarity: the foundational approach that deals with assumptions about properties of scales, multidimensional scaling that uncovers hidden structures in data, and processing-models that attempt to describe how particular tasks (e.g., identification, categorization, recognition) are carried out by the observer. In view of the central role of similarity in these three research domains, and in view of the unsolved questions about how similarity operates in a variety of perceptual tasks and conditions, the authors urge investigators in these three domains to combine their efforts to forge a general quantitative theory of similarity.

The similarity orderings of an array of objects are typically modeled by representing each object as a point or a (probabilistic) distribution of points within a space, and the particular topology assumed for the space enables similarity values to be derived for pairs of objects. Sometimes the similarity space is used simply to provide a means of efficiently predicting data from judgment tasks, and in these cases multidimensional scaling techniques can extract from a given set of data the smallest number of dimensions required to represent the similarity orderings. At other times the similarity space is regarded as a model of an (internal) psychological space, and as such it provides a means of viewing how a given collection of objects might be represented in the cognitive system. When one takes the second view, one easily slips into the more general issue of considering how internal spaces are structured. In the authors' words, "The similarity judgment format is intended only to provide a milieu for the raising of critical issues concerning psychological spaces, ..." (p. 300). It is to this broader issue that the present set of comments is directed.

The researcher who is poised to embark on the modeling of a psychological space would seem to be blessed with a reach store of mathematical structures to choose from, and the present chapter offers a fairly extensive glimpse into the range of alternative structures that have seen recent use. However, one might claim that there is an inherent disadvantage in having a large number of alternative mathematical models at hand that could satisfy a given set of psychological (or engineering) constraints (in the form of theory and/or data), and therefore additional constraints would be of benefit in narrowing the range of alternative models. Put otherwise, when a psychological space is treated in a "black box" fashion, with inputs and outputs providing the primary constraints, then the number of ways of modeling the "space" inside the black box given a particular set of input/output relations can be enormous. But if one chooses to look inside the box, one may discern structural characteristics that will usefully constrain candidate models.

Recently there has been a startling increase in the rate of new information regarding the anatomy and physiology of neural systems. Some of these findings are directly relevant to the way that we conceive of internal representational spaces and challenge the common belief that receptive fields of neurons and the topographies of networks are fixed. Probably the most well-known and vivid example of an internal representation is the map of the body surface. The typical textbook illustrations of brain maps of the body surface have fostered the notion that somatosensory cortex faithfully preserves spatial and temporal characteristics of cutaneous receptors. However, recent studies have shown that the cortical somatosensory map changes after repeated tactile stimulation (Whitsel et al., 1989) and after peripheral deafferentiation (Merzenich et al., 1983). Apparently the structure of the circuitry underlying a receptive field of a cell cluster in the cortical body map involves an excitatory center and an inhibitory surround. The receptive field of a cell or a small cluster of cells is measured by stimulating the body surface (e.g., with a camel's hair brush) and determining the surface locations that change the firing rate of that cell or cell-cluster. If a drug (bicuculline) is then injected into the cortical area to inhibit the inhibitory cells, the receptive field is found to have expanded dramatically. The basis for the expanded receptive field is attributed to the wide distribution of excitatory fibers whose extremities are normally inhibited by the overlying inhibitory fibers. The effect of inhibiting the inhibitory elements is to "unmask" the widespread projections of excitatory elements. Hence, the structure of the representational space for body locations apparently involves a broad projection of excitatory cells whose responses are locally "sculpted" by an overlying inhibitory network. Presumably, this knowledge of circuitry could be used to construct a corresponding model of stimulus generalization (perhaps along the lines of the neural network models of Shepard and Kannappan, 1991, and Staddon and Reid, 1990) of body location, which, among other things, might predict behavioral outcomes of new and perhaps more dynamic psychological task conditions suggested by the model.

The authors mention very briefly the issue of how percepts are represented in a psychological space. Typically a percept or a stimulus object has been represented in psychological space as a point, but it has been represented also as a probabilistic distribution of points. While these two ways of coding a stimulus may be appropriate for the kind of mathematical structures the authors review for the triad of foundational measurement, multidimensional scaling, and process models, in the larger issue of internal spaces, it would seem useful in model building to know how objects, attributes, and locations appear to be coded in the brain.

The area of the brain containing cells that are sensitive to discrimination and identification of objects is the inferotemporal cortex. Recent studies have shown that information sufficient for measuring discrimination capacity is not available in single neurons but only in ensembles (populations) of neurons (e.g., Gochin et al., 1992). In this cortical area the coded information about a stimulation is apparently carried independently by members of an ensemble, rather than redundantly or interactively (Gawne, Eskander, & Richmond, 1992). Other systems that have been shown to employ population coding are odor-identification cells of the olfactory bulb (Freeman & Skarda, 1985), eye-movement cells of the superior colliculus (Sparks & Mays, 1986), the limb-movement cells of the motor cortex (Georgopoulos, Swartz, & Keller, 1986), and forelimb-movement cells in the cerebellum (Pellionisz & Bloedel, 1991). It is possible for an ensemble of cells to code stimulus inputs or response outputs deterministically or probabilistically, depending on the amount of noise in the system.

In contrast to the multidimensional scaling efforts that find the minimum number of dimensions required to represent similarity orderings among a set of objects or attributes, the neural-inspired modeling of internal spaces may require a tolerance for a large number of dimensions accompanied with a search for appropriate mathematical methods for dealing with them. A related example of a mathematical model of neural internalization is given by Pellionisz and Llinas (1985), in which each neural fiber is treated as a dimension and the firing rates are treated as the dimensional values. External objects are represented first in the peripheral sensory coordinate system by a point vector (representing the electrical activity in each incoming fiber), and this sensory vector is transformed into a vector in an internal space where meaningful operations can be performed on it. A special mathematical problem arises in this transformation from an external to an internal space, because the two spaces do not in general have the same number of dimensions; that is, the number of fibers entering an internal space may be larger or smaller than the number of fibers leaving it. The type of mathematical transformation that is independent of coordinate properties is the tensor, and Pellionisz and Llinas developed a tensor network theory to describe the functions of neural circuits mathematically. It has been said that the growth of structural knowledge of neural circuits has outpaced the growth of the kinds of mathematical tools needed for adequate study of how the neural circuits might function. The modeling by Pellionisz and Llinas perhaps may be regarded as an effort to redress this imbalance.

It is desirable, from considerations of theoretical simplicity as well as efficiency of production, that the information in the (internal) representational space be invariant across a variety of tasks (e.g., identification, categorization, recognition, similarity scaling); that is, we wish the representational space to be process-independent. However, as pointed out by the authors, attention apparently can induce a transformation on the similarity orderings in a space. Consider a set of objects mapped into a similarity space based on the data obtained from an identification task. Can one use this mapping to predict performance in a categorization task, particularly under exemplar models of categorization, in which the classification of an object depends upon its similarity to objects in other categories? Apparently cross-task performance is successful for integral-dimension stimuli (for which dimensions are not readily singled out by attention), but not for separable-dimension stimuli (in which the dimensions can be relatively easily singled-out by attention). Following the earlier lead of Shepard (Shepard, Hovland, & Jenkins, 1961; Shepard & Chang, 1963), Nosofsky (1984, 1986) formulated a model of categorization in which each dimension of the similarity space is weighted such that a large attentional weight serves to "stretch" the space along the dimension while a small attentional weight serves to "shrink" the space along the dimension. Thus, on this view, the similarity spacings in a cognitive space involving separable dimensions are not invariant under changes in selective attention to particular dimensions. Variations in selective attention can occur not only across tasks, but also within a task.

The notion that attention can operate on a space by stretching or shrinking the space along a dimension may be extended to the more general notion that at least for some object or location spaces, attention distorts the space only in the neighborhood of its current focus. In particular, attention could be viewed as a local change of the space inside and outside the area of attentional focus. A mechanism that could effect this kind of local distortion on a cortical map of spatial locations is the thalamic circuit (Nicolelis, Lin, Woodward, & Chapin, 1993), and a recent study (LaBerge, Carter, & Brown, 1992) that simulated operations of this circuit in a shape-identification task showed that information flow increased in the focal area of attention relative to its surround. Since every area of the cortex is reciprocally connected to a particular thalamic nucleus containing this particular type of circuit (a cortical column typically extends downward to include its corresponding thalamic "column"), it would seem that activity in any cortical map (induced from external, bottom-up sources or from internal, top-down sources) involves this kind of local distortion of its space. The structural separation of the cortical map from its thalamic modulatory mechanism could encourage the hypothesis that the cortical map itself (during its base-rate state of cell firings) is relatively task-independent, and that task-related processes (e.g., selective attention) operate through its corresponding thalamic circuitry to produce momentary distortions in the cortical map.

Townsend and Thomas have amply demonstrated the richness of mathematical structures that are available to model psychological space, with an emphasis on the common role of similarity scaling across research domains including foundational measurement, multidimensional scaling methods, and certain cognitive process models. The present commentary serves mainly to suggest how neurobiological findings may provide helpful constraints on the choice of structures that give rise to the topologies that could represent an internal space.

Townsend and Thomas: LaBerge has keenly selected a domain of consideration which received little attention in our chapter, that of consideration of neural structure in model building and theorization. Our scant material on that subject was supplemented, at the urging of the Editor of this volume, by a few more remarks in the Conclusion on the outlook for reductionism and theory construction in general. This material was added after LaBerge had received the chapter, but in any case, really doesn't affect his commentary.

Basically, LaBerge argues that knowledge of the physiology may aid in constraining the large number of theoretical entities that are capable of producing a corpus of input-output data, to a dramatically reduced set of alternatives. As our remarks in the chapter indicate, we are entirely comfortable with this proposal. More generally, the recent fast progress in neuroscience is helping to guide many of us from day-to-day in our conception of information processing systems underlying perception and cognition.

On the other hand, it does seem that the specific goal of restraining the size of explanatory classes capable of handling well-known cognitive phenomena, is largely still in the form of a promissory note. We have not seen very many cases, for example, where an issue concerning two competing cognitive hypotheses has been decisively settled by physiological data. Rather, the primary contribution of neurophysiology has been in finding the circuitry probably responsible for cognitive processes already pretty well accepted. Even the famous Hubel and Wiesel feature work has not ruled out more continuous and global models of processing, in most researcher's minds.

Another benefaction from neuroscience has been the proliferation of specified brain regions jointly responsible for certain task elements. On the one hand, this has supported at least a weak form of modularity and on the other, has challenged both physiologists as well as psychologists to fully account for the many "centers" contributing to certain task performance.

All this being said, the future looks promising for a more sophisticated brand of interaction of neurological data with behavioral phenomena and theoretical structures. LaBerge and colleagues own recent work is striking in this regard. For instance, LaBerge, Carter, and Brown (1992), have employed current information about the visual pathways, especially vis-a-vis thalamus, to construct simulations of attentional models which obey both contemporary neural tenets as well as the general assumptions of LaBerge's model of visual attention. In our view, this kind of work could be valuable in conjunction with the quantitative tools discussed in our chapter, in building and testing more accurate theories of form cognition and similarity relations therein.

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