

**Interactive Effects of Factors Prolonging Processes  
In Latent Mental Networks**

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Suppose the subprocesses in a task are arranged in a PERT network; that is, some pairs of subprocesses are sequential but others, collateral subprocesses, are not. Sternberg showed that factors selectively affecting subprocesses in series will have additive effects. Here we present three additional results. First, factors affecting collateral subprocesses have subadditive effects. Second, if two subprocesses in series are each collateral with a third, factors selectively affecting the two serial processes have superadditive effects. Third, if two sequential subprocesses are always on the longest path through the network, factors selectively affecting them will be additive.

In 1869 Donders proposed a method whereby the durations of nonoverlapping mental subprocesses arranged serially could be assessed by subtracting the mean reaction time (RT) of experimental conditions not requiring a particular subprocess from the mean RT of conditions which required that subprocess. An extension of this method was propagated a hundred years later by Sternberg (1969). His technique was still based on serial systems but required only that the experimenter shorten or lengthen the durations of two subprocesses through manipulation of experimental factors, rather than eliminating entire contributions of the subprocesses.

The Donders method (the "subtractive method") is still used rather extensively despite its strong assumptions, but the Sternberg "additive factor method" has superseded the older approach. It too has, of course, its limitations, as well as its strengths, many of these discussed by Pachella (1974), Taylor (1976), Theios (1973), and Townsend and Ashby (1983). It is beyond the scope of this paper to develop these previous discussions in any detail, particularly since there is much theoretical and empirical material available in the literature.

One of us (e.g., Townsend, 1971, 1974, 1976; Townsend and Ashby, 1978, 1983; Ashby and Townsend, 1980) has been investigating the mathematical properties and their empirical implications of serial and parallel (i.e., subprocesses which operate simultaneously) subprocesses for a number of years. Starting at a time when almost nothing was known about parallel processing and not much about more general serial processes, he and his group focused on the mutual mimicking potential of those classes of models and on what kind of experiments might be able to test them against one another.

The durations of the subprocesses were assumed to be stochastic, that is, based on probability distributions. Most of the work until recently emphasized the processing of a set of elements (items, etc.) in parallel or in series, rather than investigating the effects of manipulating two experimental variables which separately affect two subprocesses that in turn are arranged serially or in parallel. Interestingly, it turns out that more potential is sometimes generated for testability of parallel vs serial processing in the latter case, which includes looking for additivity as is implicit in the additive factor method (Townsend and Ashby, 1983, Chapter 12; Townsend, in press). However, there has been little work on processes arranged in more complex patterns from the Townsend lab.

In contrast, Schweickert has recently proposed a new approach termed "latent network theory" (Schweickert, 1978, 1982, 1983a, 1983b, 1984). This theory is based on networks classed as PERT networks, see Figure 1. There are three main attributes of such networks. 1) If a subprocess  $X$  precedes another subprocess  $Y$  on a path, then  $X$  must be completed before  $Y$  can start. Two such subprocesses are called sequential. Note that no subprocess can begin until all its immediate predecessors have finished. 2) Two subprocesses which are not sequential can be executed in any order with respect to each other, including concurrently. Two subprocesses which are not sequential are called collateral. (Two collateral subprocesses in a PERT network are in parallel if they have the same initial vertex and the same terminal vertex.) 3) No subprocess is allowed to precede itself on a path, that is, the network is acyclic. To keep the mathematics tractable, the additional assumption that the subpro-

cess durations are fixed quantities, not random variables, was made in Schweickert (1978).

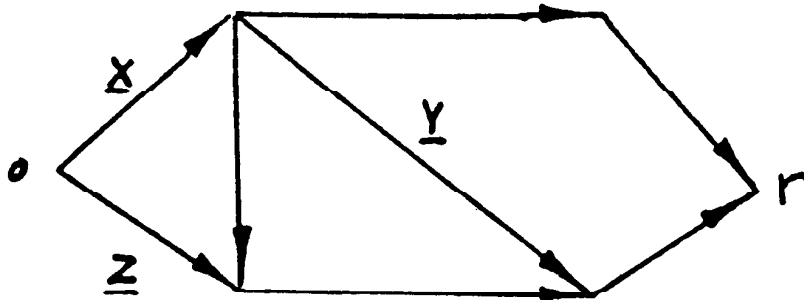


Fig. 1

A PERT network;  $X$  and  $Y$  are sequential,  $X$  and  $Z$  are collateral.

The theory is not that employed by engineers using PERT networks to, say, optimize industrial performance. In such situations the form of the network is already known. Schweickert's theory rather implements a black-box strategy of discovering the form of the unknown network. This is accomplished by selectively prolonging subprocesses as in the additive factor method, although the theory is much more general than the additive factor method.

Perhaps the major drawback at this time is the deterministic assumption, although Schweickert has made progress in assigning approximative bounds on the errors in parameter estimates when subprocess durations are random (Schweickert, 1982). Further, Fisher and Goldstein (1983) have worked out a powerful method of generating predictions involving exponential processing times.

In any event it is clear that psychology has not even scratched the

and/or can be identified in terms of their structure.

One critical facet that has begun to emerge over the years in a rather subtle fashion involves the emphasis of architecture identification procedures. Donders wanted to know the exact (average) times of particular mental subprocesses. Sternberg's technique will not permit that determination but begins to assay the structure of the architecture in that noninteractive and independent serial processes will produce RT additivity and therefore mean RT interactions falsify (within statistical error) such systems. Thus, a shift from definite numerical assignment to more qualitative aspects of a cognitive system was born.

In a similar vein and much more profoundly, the Schweickert approach is geared to discovering the manner in which the internal processes are hooked up. Likewise, Townsend and colleagues (Townsend, in press, Townsend and Ashby, 1982) discovered that a wide class of parallel systems could not predict RT additivity. A major subclass of this class (termed independent parallel models) predicts subadditivity. Basically, "subadditivity" is indicated when a mean RT increase due to one factor (say X) is smaller, the larger the mean RT is due to another factor (say Y). Superadditivity is just the opposite and, of course, "additivity" is revealed when an alteration in, say X, always causes the same change in mean RT irrespective of factor Y. Fig. 2 illustrates these prototypical characteristics.

Those results are completely general with respect to the probability distributions of the subprocesses or their parameter values if they are parameterized distributions. Thus, for instance, subadditivity rules out

serial processing and supports independent parallel processing. Conversely, additivity falsifies a substantial class of parallel systems.

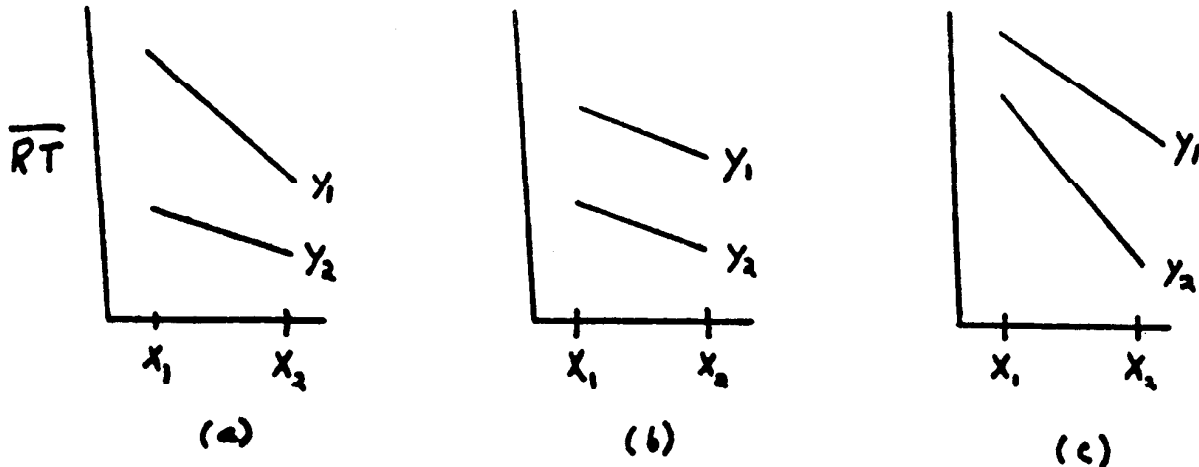


Fig. 2

Graphs illustrating (a) superadditivity, (b) additivity and (c) subadditivity. We use the convention that mean RT decreases as the factor level index increases.

We refer to this type of characteristic as a "qualitative" property of a system. It is analogous to the various possible kinds of asymptotic behavior as well as catastrophic classes of activity in very general dynamic systems.

In the present paper we initiate an advance in the qualitative structure of black-(mental)-box identification. We view the present work as an incipient step toward synthesizing Schweickert's work involving potentially highly complex but deterministic networks and Townsend's structurally simpler but stochastic systems.

Specifically, we demonstrate that two important classes of non-serial architectures exhibit nonadditive, yet qualitatively distinct types of RT behavior. The first result substantially generalizes Townsend's finding

mentioned above that independent parallel systems evince subadditivity.

The extension is to two collateral subprocesses  $X$  and  $Y$  which are embedded in a latent PERT network. Other subprocesses may proceed or follow either  $X$  and  $Y$  or both, and in fact other subprocesses may be in parallel with  $X$  and  $Y$ . It is proven that when two factors  $X$  and  $Y$  selectively influence subprocesses  $X$  and  $Y$ , then subadditivity is still entailed.

The second result concerns a system where subprocesses  $X$  and  $Y$  are arranged sequentially (i.e., either  $X$  precedes  $Y$  or vice versa) with the special case being that they are strictly serial (i.e., the end of  $X$  initiates  $Y$  or vice versa). In addition a third process  $Z$  is concurrent with  $X$  and  $Y$  or in other words is parallel with the  $X$ - $Y$  path containing  $X$  and  $Y$ , see Fig. 3. It is shown that with selective influence of the  $X$  and  $Y$  factors in a manner to be made precise below, all such systems will exhibit superadditivity.

Here we step aside to offer certain necessary definitions and notations. The reader already versed in these as given for instance in Townsend and Ashby (1983) and Schweickert (1978), may proceed to the following section.

Although there is nothing in what follows that precludes distribution functions containing jumps, the proof of theorem 2 is greatly simplified by excluding them. Thus, all distribution functions will be assumed to be absolutely continuous with respect to Lebesgue measure and strictly monotonically increasing. This corresponds to the notion that processing is going on in a continuous manner without sudden jumps or intermediate halting of the processing operation.

Let  $f_X(t;X,Y)$  be the density function on positive (or zero) processing times for process  $X$  and similarly for process  $Y$ . Specific values of  $f$  for given levels  $x$  and  $y$  of the factor  $X$  and  $Y$  will be denoted by  $f_X(t;x,y)$ . Note that in principle this density may depend on the  $Y$  experimental factor as well as the  $X$  factor. The distribution function will be denoted  $F_X(t;X,Y)$  and so on for the other subprocesses. The definitions will be given in terms of the  $X$  subprocess; those for  $Y$  are exactly the same.

Definition 1

Subprocess  $X$  is selectively affected by experimental factor

$X$  if and only if  $f_X(t;X,Y) = f_X(t;X)$  for all  $t > 0$ .

Thus, selective influence is reasonably defined by a functional independence from the other factor, in this case  $Y$ . We further assume that the experimental factors  $X$  and  $Y$  have been arranged so that increasing a factor tends to decrease processing duration (i.e., increase speed) in some suitably defined fashion. Townsend and Ashby (1978, 1983) have discussed various ways in which factors could affect distributions and these vary in terms of the strength of their implications. For instance, increasing  $X$  might diminish the mean of the  $X$  distributions; it might order the distribution functions in the sense that if  $x_1 > x_2$  then  $F_X(t;x_1) > F_X(t;x_2)$ ; the hazard functions might be ordered; the likelihood ratio might be monotone. In fact, the last condition implies the next to last, back to the first which is the weakest; so the ordering of the means is not a very strong statement about the effects of the factors. The ordering of the distribution functions is also implied by the condition that the density



functions  $f_X(t;x_1)$  and  $f_X(t;x_2)$  cross only once (for  $t > 0$ ); which is in turn implied by the condition that  $X$  simply act as a shift parameter, that is,  $f_X(t;x+h) = f_X(t+r(h);x)$  where  $r$  is a function of  $h$  but is constant once  $h > 0$  is fixed. (Note that the latter is usually inappropriate for RT densities which must be defined for  $t \geq 0$ ).

The one point crossover condition seems like a reasonable place to begin, relatively strong but stronger than an ordering on the hazard functions (see Townsend and Ashby, 1983 p. 283). The second result is based on this type of influence although we later forecast the possibilities for generalizing the theorem to weaker kinds of influence.

Fig. 3 and 4 show the mental networks with which we shall work here. Figure 3 exhibits a typical structure to which Theorem 1 applies, and Figures 4a and 4b illustrate the networks to which Theorem 2 applies. Observe that subprocesses  $\underline{X}$  and  $\underline{Y}$  in Figure 3 are concurrent but not parallel because they do not ordinarily begin at the same instant.

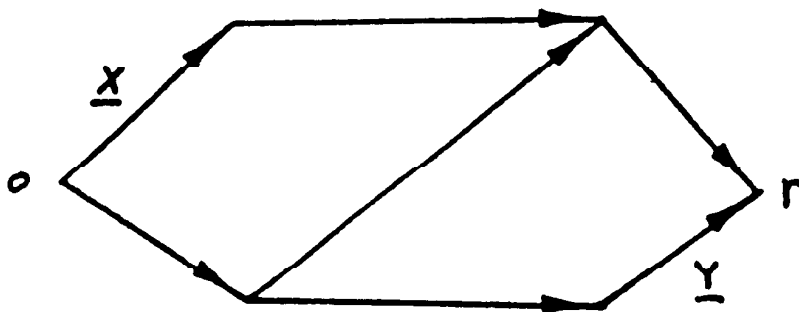


Fig. 3

A PERT network with  $\underline{X}$  and  $\underline{Y}$  collateral.

## Collateral Subprocesses Yield Subadditivity

In the first theorem we show that under plausible assumptions, experimental factors prolonging collateral subprocesses in a stochastic PERT network will have subadditive effects on reaction time. Subprocesses  $X$  and  $Y$  in Fig. 3 are collateral. Note that  $X$  need not be executed before  $Y$ , nor  $Y$  before  $X$ ; the subprocesses are free to be executed concurrently.

We suppose the experimental factor  $X$  has two levels,  $x_1$  and  $x_2$ , and affects subprocess  $X$ . Let  $T_{x_1}$  and  $T_{x_2}$  be random variables corresponding to the durations of  $X$  when factor  $X$  has levels  $x_1$  and  $x_2$ , respectively. Likewise, experimental factor  $Y$  with levels  $y_1$  and  $y_2$  affects subprocess  $Y$ . Let  $T_{y_1}$  and  $T_{y_2}$  denote the duration of  $Y$  when  $Y$  has levels  $y_1$  and  $y_2$ , respectively. Finally, let  $T(x_1, y_1)$  be the time to complete the task, i.e., the reaction time, when  $X$  has level  $x_1$  and  $Y$  has level  $y_1$ . Let  $t(x_1, y_1)$  denote a value taken on by  $T(x_1, y_1)$ . Other expressions are defined in the analogous way.

Theorem 1.

Let  $X$  and  $Y$  be collateral subprocesses selectively affected by factors  $X$  and  $Y$ , respectively. Suppose changing the level of  $X$  from  $x_2$  to  $x_1$  increases the duration of subprocess  $X$  in the sense that

$$T_{x_1} = T_{x_2} + \Delta T_x,$$

where  $\Delta T_x$  is a nonnegative random variable.

Likewise,

$$T_{y_1} = T_{y_2} + \Delta T_y,$$

where  $\Delta T_y$  is a nonnegative random variable.

Let  $S = \{ \langle t_{x_1}, t_{y_1}, \Delta t_x, \Delta t_y \rangle \}$  be the space of values taken on by  $T_{x_1}$ ,  $T_{y_1}$ ,  $\Delta T_x$  and  $\Delta T_y$ , respectively. Suppose there is a region  $R$  contained in

S where

$$t(x_2, y_2) - t(x_2, y_1) < 0$$

and  $t(x_2, y_2) - t(x_1, y_2) < 0$

when  $T_{x_1}, T_{y_1}, \Delta T_x$  and  $\Delta T_y$  take on values in  $R$ .

Suppose  $P(R) = P(\langle t_x, t_y, \Delta t_x, \Delta t_y \rangle \in R) > 0$ .

Then factors  $X$  and  $Y$  have subadditive effects.

Proof:

The proof depends on the following result, established in Schweickert (1978). For given values  $t_{x_2}, t_{y_2}, \Delta t_x$  and  $\Delta t_y$  where  $t_{x_1} = t_{x_2} + \Delta t_x$  and

$$t_{y_1} = t_{y_2} + \Delta t_y,$$

$$\begin{aligned} t(x_1, y_1) &= \max \{t(x_1, y_2), t(x_2, y_1)\} \\ &= t(x_1, y_2) + t(x_2, y_1) - \min \{t(x_1, y_2), t(x_2, y_1)\}. \end{aligned}$$

$$\begin{aligned} \text{So, } \Delta^2 &= t(x_1, y_1) - t(x_1, y_2) - t(x_2, y_1) + t(x_2, y_2) \\ &= t(x_2, y_2) - \min \{t(x_1, y_2), t(x_2, y_1)\} \\ &= \max \{t(x_2, y_2) - t(x_1, y_2), t(x_2, y_2) - t(x_2, y_1)\}. \end{aligned}$$

Let  $a = t(x_2, y_2) - t(x_1, y_2)$  and  $b = t(x_2, y_2) - t(x_2, y_1)$ .

In the region  $R$ ,  $a < 0$  and  $b < 0$ .

$$\begin{aligned} \text{Then } E\Delta^2 &= E \max \{a, b\} \\ &= 0[1 - P(R)] + E \max \{a, b \mid a < 0 \text{ and } b < 0\} P(R) < 0. \end{aligned}$$

End of proof.

When Two Subprocesses are SERIAL and  
Both are Concurrent with another Subprocesses

The class of networks under consideration in this section is given in Fig. 4. The prototype is in Fig. 4a and a typical member of the general class is in Fig. 4b.

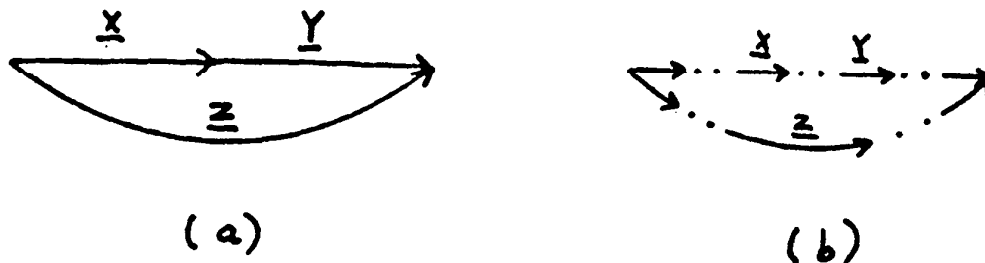


Fig. 4

- (a) Subprocesses  $X$  and  $Y$  in series, both collateral with  $Z$ .  
 (b) An equivalent network.

First note that there may be any number of other subprocesses in series with  $X$  and  $Y$  and any number of other subprocesses that are concurrent with the  $X$ - $Y$  path. However, we do exclude other paths which include  $X$  but not  $Y$  and so on. It is quite obvious that it is legitimate to collate the processes preceding  $X$  together with  $X$  itself and simply call the collection  $X$ . Similarly we can collate  $Y$  together with those processes following it and call the whole collection  $Y$ . Too, we may simply consider a new distribution on whatever else is concurrent with the  $X$ - $Y$  path and simply refer to it as the  $Z$  random variable.

As in Theorem 1, we assume that the  $X$ ,  $Y$  and  $Z$  processes are probabilistically independent. Furthermore, it is postulated that selective influence makes itself felt at least at the level of ensuring that

when, say, factor  $X$  is changed from  $x_2$  to  $x_1$  that the two  $X$  densities cross exactly once; that is, for  $t > 0$  there exists just one  $t=t^*$  such that  $f_X(t^*;x_1) = f_X(t^*;x_2)$ . For  $t < t^*$ ,  $f_X(t;x_2) > f_X(t;x_1)$  and when  $t > t^*$  then  $f_X(t;x_2) < f_X(t;x_1)$ . As mentioned in the introduction, this serves to order the distribution functions  $F_X(t;x_2) > F_X(t;x_1)$  as well as the mean processing times but this assumption is not strong enough to order the hazard functions. In subsequent work we usually will simplify the notation to  $f_X(t;x_2) = f_{X_2}(t)$  and so on.

Recall that mean RTs are superadditive if the difference between mean RTs for a difference on one factor increases, the smaller the other factor. This is intuitively because when one factor is very large, the overall RT is quite fast and tends to be obscured by the concurrent  $Z$  sub-process. When that factor decreases, the overall  $X$ - $Y$  path is pushed up in duration in contention with the  $Z$  path and the other factor makes itself felt more strongly.

We are now in a position to prove Theorem 2, which signifies superadditivity for a broad class of processes in a nonparametric fashion.

### Theorem 2

Consider a PERT network consisting of subprocesses  $X$ ,  $Y$  and  $Z$ . Suppose  $X$  and  $Y$  are in series with, say,  $X$  preceding  $Y$ . Suppose the initial vertex of  $Z$  is the same as the initial vertex of  $X$ , and the terminal vertex of  $Z$  is the same as the terminal vertex of  $Y$  (see Fig. 4a). Suppose the durations of  $X$ ,  $Y$ , and  $Z$  are stochastically independent.

Suppose experimental factors  $X$  and  $Y$  selectively affect sub-

processes  $X$  and  $Y$ , respectively, each factor producing the cross-over condition. Then as  $X$  and  $Y$  cause prolongations of the random times  $t_x$  and  $t_y$ , respectively, in the cross-over sense, the mean RTs are superadditive.

Proof:

Let us express the second order difference on mean (expected) RT for a specific pair of increments on

$X$  and  $Y$ ,  $(x_1, x_2)$  and  $(y_1, y_2)$  as

$$\Delta^2 E(T; X, Y) = E(T; x_1, y_1) - E(T; x_1, y_2) - [E(T; x_2, y_1) - E(T; x_2, y_2)]. \quad (1)$$

Thus, we are expressing the second order difference in mean RTs in the direction of increasing RT and decreasing factor values  $x$  and  $y$ . If  $\Delta^2$  is  $>0$  we have superadditivity; if  $\Delta^2 < 0$  subadditivity characterizes the network; if  $\Delta^2 = 0$  additivity is the result. It is well known that for a non-negative valued variable  $T$ ,

$$E(T) = \int_0^{\infty} 1 - F(t) dt, \text{ where } F \text{ is the distribution function on } T.$$

In the present situation, the overall RT, represented by  $T$ , is the maximum of the durations of the serial path containing  $X$  and  $Y$  and the other path  $Z$  which is in parallel with the former. Therefore, by independence,

$$P(t; X, Y) = P(\text{Max}(T_x, T_x + T_y) < t) = F_z(t) P(T_x + T_y < t).$$

$$\text{Further, } P(T_x + T_y < t) = f_x * f_y = \int_0^t f_x(u) f_y(t-u) du.$$

It is therefore feasible to represent the second order difference Eq.(1), first symbolically for intuition and then rigorously as

$$\begin{aligned} \Delta^2 E(T, x, y) &= E_{11} - E_{12} - E_{21} + E_{22} \\ &= \int (1-f_{x_1} * F_{y_1} F_z) - \int (1-f_{x_1} * F_{y_2} F_z) \\ &\quad - \left[ \int (1-f_{x_2} * F_{y_1} F_z) - \int (1-f_{x_2} * F_{y_2} F_z) \right] \end{aligned} \tag{2}$$

which by linearity of integration and convolution equals

$$\begin{aligned} &= \int (f_{x_1} - f_{x_2}) * (F_{y_2} - F_{y_1}) F_z \\ &= \int_{l=0}^{\infty} \int_{u=0}^t (f_{x_1}(u) - f_{x_2}(u)) (F_{y_2}(t-u) - F_{y_1}(t-u)) F_z(t) dt \end{aligned} \tag{3}$$

It will aid our purpose to simplify our functions further.

So, letting  $t^*$  be the cross-over point for  $f_x$  we set

$$a_1(t) = \begin{cases} f_{x_2}(t) - f_{x_1}(t), & 0 \leq t \leq t^* \\ 0 & t^* < t \end{cases}$$

$$a_2(t) = \begin{cases} f_{x_1}(t) - f_{x_2}(t), & t^* < t < +\infty \\ 0 & t < t^* \end{cases}$$

Observe that  $a_1(t) > 0$  for  $t < t^*$ , and is zero thereafter and

that  $a_2(t) > 0$  for  $t > t^*$ , and is zero elsewhere.

We further set  $F_{y_2}(t) - F_{y_1}(t) = b(t)$  and  $F_z(t) = c(t)$ .

Notice that  $b(t)$  is always positive except at  $t=0$  and in the limit

as  $t \rightarrow \infty$ . Using this notation we may reexpress Eq.(3) as

$$\begin{aligned}
 -\Delta^2 E(T, x, y) &= \int_{t=0}^{\infty} \int_{u=0}^t a_1(u) b(t-u) du c(t) dt \\
 &- \int_{t=0}^{\infty} \left[ \int_{u=0}^t a_2(u) b(t-u) du \right] c(t) dt.
 \end{aligned} \tag{4}$$

The nice aspect of Eq.(4) is that the term involving  $a_1$  contributes all the positive weight to the overall integral whereas the term involving  $a_2$  contributes all the negative weight.

Our next project is to render the  $a_1$  and  $a_2$  terms more comparable in order to assess whether the difference illustrated in Eq.(4) is positive, negative or zero. We do this primarily by operating on the  $a_1$  and  $a_2$  term

By the cross-over condition,

$$\begin{aligned}
 &\int_{t=0}^{\infty} \left[ \int_{u=0}^t a_2(u) b(t-u) du \right] c(t) dt \\
 &= \int_{t=t^*}^{\infty} \left[ \int_{u=t^*}^t a_2(u) b(t-u) du \right] c(t) dt.
 \end{aligned} \tag{5}$$

Now set  $u = u' + t^*$  to transform (5) into

$$\int_{t=t^*}^{\infty} \left[ \int_{u'=0}^{t-t^*} a_2(u'+t^*) b(t-u'-t^*) du' \right] c(t) dt$$

and afterward, define  $a_2^1(u') \equiv a_2(u'+t^*)$  to get Eq.(5) equal to

$$\int_{t=t^*}^{\infty} \left[ \int_{u'=0}^{t-t^*} a_2^1(u') b(t-u'-t^*) du' \right] c(t) dt.$$

Finally, let  $t' = t - t^*$  to achieve

$$\int_{t'=0}^{\infty} \left[ \int_{u'=0}^{t'} a_2^1(u') b(t'-u') du' \right] c(t'+t^*) dt' \tag{6}$$



The negative term is now in the same form as the positive term in that  $a_2(t)$  starts at  $t=0$  and is weighted by  $b(t-u)$  (neglecting the primes which may now be disposed of). An important difference is that we see  $c(t+t^*)$  rather than  $c(t)$  as in the  $a_1$  term. This will have major consequences.

By Dirichlet's change of order of integration formula,

$$\int_{t=0}^{t_1} \left[ \int_{u=0}^t f(u,t) du \right] dt = \int_{u=0}^{t_1} \left[ \int_{t=u}^{t_1} f(u,t) dt \right] du$$

and taking the limit of either side as  $t_1 \rightarrow +\infty$  yields the result

we need. First we operate on the  $a_1$  term:

$$\begin{aligned} & \int_{t=0}^{\infty} \left[ \int_{u=0}^t a_1(u)b(t-u)du \right] c(t)dt \\ &= \int_{u=0}^{\infty} \left[ \int_{t=u}^{\infty} a_1(u)b(t-u)du \right] c(t)dt \\ &= \int_{u=0}^{\infty} a_1(u) \left[ \int_{t=u}^{\infty} b(t-u)c(t)dt \right] du \end{aligned} \tag{7}$$

Now, set  $t' = t-u$  and recall that  $a_1(t)$  is non zero only for  $0 \leq t \leq t^*$ .

This permits the following transformation of Eqn(7).

$$\begin{aligned} & \int_{u=0}^{\infty} a_1(u) \left[ \int_{t=u}^{\infty} b(t-u)c(t)dt \right] du \\ &= \int_{u=0}^{t^*} a_1(u) \left[ \int_{t'=0}^{\infty} b(t')c(t'+u)dt' \right] du \\ &= \int_{u=0}^{t^*} a_1(u) \left[ \int_{t'=0}^{\infty} b(t')c(t'+u)dt' \right] du \\ &< \int_{u=0}^{t^*} a_1(u) \left[ \int_{t'=0}^{\infty} b(t')c(t'+t^*)dt' \right] du \quad \left[ \text{Because } u \leq t^* \right] \\ &= \left[ \int_{u=0}^{t^*} a_1(u)du \right] \left[ \int_{t=0}^{\infty} b(t)c(t+t^*)dt \right] \end{aligned} \tag{8}$$

Turning to the other, negative contribution of Eq.(6), we see that

$$\begin{aligned} & \int_{u=0}^{\infty} a_2'(u) \left[ \int_{t=u}^{\infty} b(t-u)c(t+t^*)dt \right] du \\ & > \int_{u=0}^{\infty} a_2'(u) \left[ \int_{t=u}^{\infty} b(t-u)c(t+t^*-u)dt \right] du \end{aligned} \quad (9)$$

and setting  $t' = t-u$  the right hand side of inequality(9) becomes

$$\begin{aligned} & \int_{u=0}^{\infty} a_2'(u) \left[ \int_{t'=0}^{\infty} b(t')c(t'+t^*)dt' \right] du \\ & = \left[ \int_{u=0}^{\infty} a_2'(u)du \right] \left[ \int_{t'=0}^{\infty} b(t')c(t'+t^*)dt' \right] \end{aligned} \quad (10)$$

Quite obviously the right hand multiplicand in the last formula of Eq.(8) is identical to the corresponding term in Eq.(10).

Not quite so obviously, the left hand terms in the products of Eqs.(8) and (10) are also identical. That is because

$$\begin{aligned} \int_{u=0}^{\infty} a_2'(u)du &= \int_{u=t^*}^{\infty} a_2(u)du = \int_{u=t^*}^{\infty} \left[ f_{x_1}(u) - f_{x_2}(u) \right] du = 1 - F_{x_1}(t^*) - (1 - F_{x_2}(t^*)) \\ &= F_{x_2}(t^*) - F_{x_1}(t^*) \end{aligned}$$

whereas

$$\int_{u=0}^{t^*} a_1(u)du = \int_{u=0}^{t^*} \left[ f_{x_2}(u) - f_{x_1}(u) \right] du = F_{x_2}(t^*) - F_{x_1}(t^*)$$

Call the right hand sides of Eqs.(8) and (10) A. Then by the inequalities (8) and (9) we ascertain that (Positive contribution) < A < (Negative contribution). Thus, the negative contribution is larger than the positive one and the overall result is negative. But we are scrutinizing  $-\Delta^2 E(T;X,Y)$ , so the overall second order difference is strictly positive and therefore the system is superadditive in mean RT. End of proof.

Sequential Subprocesses Yield Additivity  
When their Durations are Long.

The preceding theorem shows that factors affecting sequential subprocesses in a certain network have nonadditive effects. We suspect that factors have additive effects only in very special cases, such as when the subprocesses are serial, or when the network approaches a serial network in the limit because the subprocesses X and Y durations are long to start with. The following theorem applies in the latter case.

Theorem 3

Suppose subprocesses X and Y are sequential and are selectively affected by factors X and Y, respectively. Suppose X and Y are always on the longest path from o, the start of the network, to r, the terminus of the network, for all levels of the factors. Then the factors X and Y have additive effects.

Proof: Let the initial vertex of subprocess X be denoted X' and the terminal vertex X". Suppose, without loss of generality, that X precedes Y. Let T(o, X') denote the duration of the longest path from o to X'. Other expressions are defined in the analogous way.

When X and Y are both on the longest path from o to r, the reaction time is the duration of this longest path, which is

$$T(o, x') + T^X + T(X'', Y') + T^Y + T(Y'', r).$$

The result follows immediately. End of proof.

Discussion

We have demonstrated that two broad classes of latent mental networks exhibit opposite qualitative behaviors and neither is able to predict mean RT additivity as a function of two experimental factors.

We view the present results as a small but significant step toward a theory relating more complex stochastic networks to important qualitative characteristics of RT. Ultimately, we would hope that we and others will expand this type of exploration to include very broad classes of networks. We anticipate that for a given qualitative characteristic, for instance superadditivity, there will exist a canonical subclass of networks which predict the characteristic while all others do not. As more dependent variables such as accuracy are studied and the effects of n-tuples of factors are brought into play, we expect that the canonical class capable of predicting a particular characteristic will diminish in size, thus enhancing identification of the latent architecture involved in a cognitive task.

At the risk of pushing our luck, the above results combined with those of Townsend and Ashby (1983, chapt. 12) and Townsend (in press) suggest to us that mean RT additivity is a rare kind of bird and may be confined to serial systems, or to systems that approach a serial system in the limit as processes are prolonged. Our final fling with soothsaying is to anticipate that for networks of moderate to complex structure, sub-super- and plain additivity will be found in different regions of the domains of factors X and Y.

We suspect that Theorem 2 can be generalized to the weaker condition that the distribution functions are ordered. However, another speculation is that an ordering of the means of a distribution by the experimental factors will be too weak to imply superadditivity. This also has not yet been proven.

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