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Approach–Avoidance: Return to Dynamic Decision Behavior

James T. Townsend
Jerome R. Busemeyer
Purdue University

The purpose of this chapter is to resurrect an old area of research in psychology, that of the study of "approach–avoidance behavior." This research topic, more or less forgotten for some time, can perhaps serve as a linkage joining traditional motivation and learning psychology to the field of decision making. In fact, we suspect that approach–avoidance characterizes most, if not all, of a person's life decisions. "Borrowing" a cookie from Grandma's cookie jar, the dilemmas of dating and marriage proposals, college military duty, applying for jobs, and confronting the schoolyard bully are a tiny sample of such instances, which the reader can undoubtedly complement many times over.

But why disinter a research problem presumably "laid to rest?" Psychology, even in the more rigorous domains, is renowned for its faddish inclinations. Novel research problems are unearthed and mined by the innovator for the "obvious" nuggets and then as difficulties or complexities begin to appear, they tend to be abandoned for new "mother lodes." Researchers who, in the spirit of the hard sciences, continue to investigate the phenomena often meet with obstacles in publishing their results, because the research topic is "out-of-date." It is not difficult to see that sustained scientific progress is slow when pursued in this fashion. It is made worse by the inadequate treatment of classic and historic research accorded our students in many of our undergraduate and graduate programs.

In the present instance, we have on the one hand a lacuna in the formal theories of decision making and on the other, a fruitful area that formerly provided a link among motivation, learning, and decision making behavior. Formal theorization in decision making has been virtually dominated by utility theory and statistical decision theory, especially as put forth by Von Neumann and Morgenstern and advanced later by Savage and others. Much beautiful mathematical work has been accomplished in these pursuits, especially in statistical decision theory, game theory, and of course, subjective expected utility theory.

There are many advantages of these approaches, apart from their elegance and employment in telling us what our optimal behavior should be like and their possible use in artificial intelligence. One well known justification is to provide a touchstone with which to compare the not-so optimal behavior of real life creatures. This concept has proven of great value in certain areas, some related to decision making. The best known in experimental psychology is probably signal detection theory (e.g., Green & Swets, 1966). It is also fair to say that this rationale has been beneficial in decision research per se, in pointing out how human decisions depart from the more ideal assumptions or theorems of utility theory.

A missing quality, from the present viewpoint, are natural connections with biology and more fundamentally, with that heritage in psychology built on evolutionary principles. Growing out of William James and others' "functionalistic" characterizations of motivated behavior (i.e., organisms tend to do what helps them survive; close to a tautology, but of immense importance in bringing biology and psychology closer together), later "dynamic" psychologists (e.g., Angell, Woodworth, etc.,) began to forge a true psychology of motivation. Motivation went on to underpin a great deal of the theory and experiment in learning, as expressed primarily in behavioristic animal research for more than half a century (e.g., Hull, Tolman, Skinner, etc.). A valid criticism of behaviorism is that it tended to depreciate central cognition and another is that it ironically led to studying behavior in animals that was less than naturalistic with respect to their original environment. The field of ethology (e.g., Tinbergen, Lorenz, etc.) has done much to redress this latter skew. The emergence of "animal cognition" has also aided in this regard (e.g., see Roitblat, Bever, & Terrace, 1984). Nevertheless, the classical behaviorists left us a legacy of ideas that even now influence our ways of thinking about human and animal behavior.

With regard to utility theory, there often seems to be an absence of what underlying motives may drive the utility that an object or act may hold for a person. A number of other consequences tend to follow in the wake of the spirit of utility theory that carry it further from human activities. For instance utilities are usually conceived as static summaries of a fixed set of independent dimensional features or attributes. In reality these are, to the extent that they do exist, probably dynamically changing over time.

Other recent developments related to utility theory have seen a substantial increase in concern with what people actually do. This has been evident in empirical research partly stimulated by testing various tenets and consequences of utility theory (see, e.g., Tversky & Kahneman, 1981). One outcome has been [illegible] of more psychologically attuned qualitative or computer simulation theories and models (e.g., Johnson & Payne, 1985). Another has been the attempt to relax certain of the assumptions of utility theory in order to better accommodate human decisional frailties. As examples, we cite Fishburn (1986), Luce and Narens (1985) and perhaps with most impact on empirical psychology, Kahneman and Tversky (1979). Kahneman and Tversky (1979) have gone far in providing intriguing evidence of humankind's straying from utility theory's predictions. In doing so, they continue a tradition starting as far back as 1738 with Bernouilli's treatment of the Petersburg paradox and later with the paradoxes of Allais (1953) and Ellsberg (1961).

Finally let us emphasize that our intent is not to denigrate the field of utility theory and the rich heritage it has brought us, but rather to help fill in the mostly unoccupied and (we feel) neglected regions where traditional psychology and biology may have something useful to say about decision making.

LEWIN'S FIELD THEORY AND RELATED NOTIONS OF APPROACH-AVOIDANCE

An "old" area of research that effectively captures the importance of dynamic conception of utilities, motivation and dynamism is so-called approach-avoidance behavior. As a twentieth-century topic, its prime and likely earliest promulgator seems to have been Lewin (1935). Lewin's theory was an example of a gestalt field theory with an emphasis on spatial relations among psychological objects, as opposed to the presumably more independent congeries of "atoms" associated with British empiricism, structuralism and to some extent behaviorism (see, e.g., Boring, 1957). The other major field "theory" (neither was formalized to any extent) was Köhler's brain field (e.g., Köhler & Held, 1949). Both flow out of gestalt tradition (e.g., Hilgard, 1956).

Lewin characterized the human as moving in a space replete with psychological and physical objects possessing attractive or repulsive qualities that would tend to draw the person toward or away from the object. Objects could also have both positive and negative qualities at the same time, resulting in ambivalence on the part of the person. The positive and negative "charges" were called "valences." In order to escape the narrow confines of geometry, which Lewin thought too confining and inapplicable, he drew on the concepts of point-set topology. Topology is that mathematical field that relinquishes such devices as angle, orthogonal dimension, linear order, and so on in order to learn what may be preserved by continuous functions of one topological space to another, without the usual assumptions of Euclidean metric and the like. Lewin apparently did not pretend to be a mathematician and was widely believed to be a genius by his followers.

On rereading some of his early works (e.g., Lewin, 1936), we expected to uncover serious misapprehensions of mathematical concepts. Surprisingly little to grouse about was found, although as noted earlier, Lewin did not himself use any actual mathematics in his theorizing. What he did was to develop metaphors based on the topological notions.

It is an open question as to how much of the actual topology will turn out to be useful in experimental arenas. From our initial perspective, it appears easier to get somewhere when we spruce up the spaces to the level of differentiable vector spaces. For then we can talk about direction, speed and related notions. We will return to Lewinian ways of depicting the psychological space, but first must fill in more of the history of approach–avoidance research.

It was not long after Lewin began to publish articles and books on dynamic psychological spaces that Clark Hull incorporated the concept of approach–avoidance in his own theory (e.g., Hull, 1938; excellent discussions of Hull's and Lewin's theories may be found in Hilgard, 1956 and in chapters in the anthology by Koch on Hull, and Estes on Lewin, *Modern Learning Theory,* 1954). It was in fact a natural extension of his ideas on drive, learning, response strength, inhibition, and extinction. Lewin began at the relatively grandiose level of complex human motivation and social movement and interaction whereas Hull started with fairly detailed (though not always entirely rigorous) assumptions at a quite microlevel of psychological processes. Nevertheless Hull intended the theoretical consequences to be applicable at least in principle to more complex human pursuits; although he never seemed overly interested in social interactions per se, in contrast to Lewin. Further, Hull used some real mathematics, although it was rather more down to earth than the topology to which Lewin verbally referred. He went so far as to define a simple differential equation for the separate approach and avoidance as functions of distance from the specified object, hereafter called the "goal object" or simply "goal." This led to exponential types of valence or gradient curves that yielded qualitative predictions for experimental situations. However, Hull never got around to linking up the approach and avoidance curves to dynamic differential equations *in time* (i.e., Hull's expressions were differential equations in space rather than time). It is interesting that during roughly the same period, Rashevsky and others were using time-differential equations (i.e., with time as an independent variable) to describe hypothetical psychological and neuropsychological systems. However, there seems to have been little or no cross-talk between these groups.

The next major figure in the short history of the subject was Neal Miller, a student of Hull. Miller (1959) and Dollard and Miller (1950), applied the ideas of approach–avoidance and other facets of a neobehavioristic theory to many aspects of psychology, with special emphasis on clinical implications. The quandary posed by a phobia such as acrophobia (fear of high places) for a person living in the age of space and flight is obvious, and qualitatively well described by approach–avoidance notions. For some reasons, after a major summary of progress to date published in Volume 2 of the influential

Psychology: A Study of a Science series (1959), Miller seems largely to have moved to other interests.

It seems strange that through all the years that concepts associated with positive and negative aspects of objects (used as is typical here, in a general sense) have been bruited about, no one has ever developed a dynamical mathematical theory to capture the essentials of Lewin's verbal and Hull's slightly more rigorous ideas. In this chapter, we begin the development of a formal theory of decisional conflict behavior, which we believe characterizes most, if not all of the major decisions in a human life. We hope also to soon begin experimentation with human subjects employing sufficiently potent positive and negative rewards that at least a modicum of realism is attained.

Our development treats positive and negatively valenced influences in a descriptively simple fashion. The exact mechanisms, psychological or biological, are mostly unspecified at this point. This is not meant to underplay the potential great importance of mental factors, such as personal feelings of efficacy, for which, for example, Bandura and his colleagues (e.g., in press) have provided ample evidence in motivated behavior. We nevertheless believe (although it is probably unprovable), that various mental facets of approach–avoidance behavior evolved over eons from reflex-like acts to more internalized cognitive behavior such as mental vicarious trial-and-error routines. Interactions or even subservience to other subprocesses such as personality and efficacy perhaps also emerged over the course of evolution.

Figure 5.1 shows the prototypical type of diagram devised by Lewin where we see, in this instance, a person (P) propelled by positive drive toward a goal but repelled by the negative aspects of an intermediary task, a frequent enough situation in real life. This is an example of single goal, approach–avoidance. We use the term "goal" even in cases where all valence is aversive, for simplicity of expression. Of course, just as ubiquitous are situations where there exist unpleasant consequences that arise after a pleasant goal has been reached. Notably too, the classical gambles of uncertain decision making fall readily into the framework. In particular, risk may be viewed as adding to potential aversive consequences and thereby feeding the avoidance motivation, with gain or winnings providing for approach motivation (cf. Coombs & Avrunin, 1977). Of course, the situation becomes more complex in the presence of risk-seeking people but that does not lessen the importance of the motivational distinction between the positive and negative aspects of a situation.

Figure 5.2 shows a situation in which both of two goals are aversive. In this type of situation, given an opportunity, the person may "leave the field" so to speak, as suggested by the dotted line in the Figure. However, if the dimensionality of the field is too restricted, then that may not be possible as we will see, and the individual may vacillate or come to rest at an equilibrium point.

Even in the absence of other more positive objects in the vicinity of the aversive choices, the rest of the field away from those despised objects is

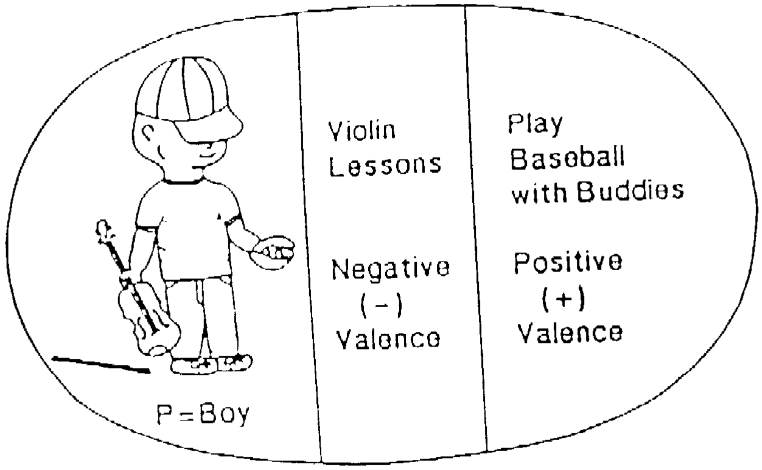

FIG. 5.1. Example of a single-goal approach–avoidance situation when the boy (*P*) is attracted toward playing baseball(+) but repelled by a violin lesson(−) that stands between him and the attractive goal.

positive relative to the aversive negative goals. Of course, there is an infinitude of rich possibilities of such motivational fields. In this chapter, we deal with but a few of the simplest, but in a rigorous way.

Miller's verbal postulates for approach–avoidance seem a good place to start a discussion that leads to a more formal system.

1. The tendency to approach a goal is stronger the nearer the subject is to it. This is an application of Hull's principle of the goal gradient and will be called the *Gradient of Approach.*

2. The tendency to avoid a feared stimulus is stronger the nearer the subject is to it. This was an extension of the general idea of the gradient of reinforcement to avoidance learning. It will be called the *Gradient of Avoidance.*

3. The strength of avoidance increases more rapidly with nearness than does that of approach. In other words, the gradient of avoidance is steeper than that of approach. This was a new assumption necessary to account for the behavior of going part way and then stopping.

4. The strength of tendencies to approach or avoid varies directly with the strength of the drive upon which they are based. In other words, an increase in drive raises the height of the entire gradient. This assumption was necessary to explain the fact that stronger shocks stopped the animals whereas weaker shocks did not and also to explain the intuitively expected result that stronger shocks would be necessary to stop hungrier animals. This assumption was a specific application of the general notion that response strength varies with relevant drive.

5. Below the asymptote of learning, increasing the number of reinforced trials will increase the strength of the response tendency that is reinforced.

6. When two incompatible responses are in conflict, the stronger one will occur.

Miller goes on to form a set of deductions from the postulates. However the majority of the "deductions" actually seem to be more like assertions about the relations among the independent variables (deprivation, stimulus intensity, training) and what corresponds to parameters in the approach–avoidance model (e.g., slopes and intercepts of the gradients). Once the model is formalized, these turn out to be obvious consequences. Our main goal is rather to deduce relations between the dynamics of behavior (movement toward or away from goals and vacillation) and model parameters.

Hull (e.g., 1938) captured the increases of both positive and negative tendencies as the organism nears a goal conveniently set at the origin, by the spatial differential equation $dE/dD = -b/D$ where $E=$ excitatory potential (i.e., tendency toward a response), $D=$ distance from goal, and b is a constant of proportionality. This obviously leads to a solution of the form $E=a-b\ \log(D)$. Another suggested form for E was $E=a\ \exp(-hxD)$. In the former case, E evidently becomes infinitely large as one approaches the goal whereas it stays finite in the second form. The first case of E approaching infinity is not necessarily a major consideration because physics has many similar examples; for instance, the force between two bodies is said to be proportional to the squared distance between them. Thus, if the two bodies approach one another, the force ideally should approach infinity—at the time of impact, of course, a discontinuity is introduced into the dynamics. Miller (1959) pictured the gradients as linear, although explicit formulae were not given. We will later compare some different approach–avoidance curves, or "gradients" as they have traditionally been called. It is likely that

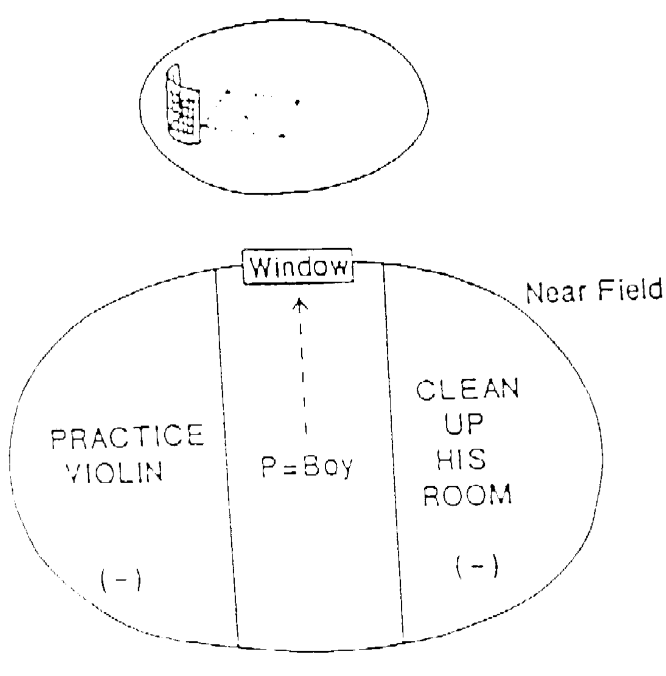

FIG. 5.2. Example of a double-goal avoidance–avoidance situation where the person is repelled by both goals and may leave the field (dashed arrow).

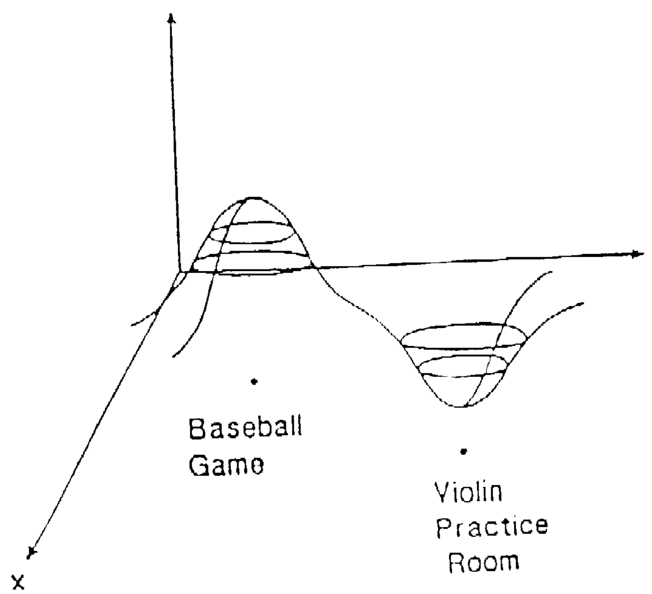

FIG. 5.3. An example of a double-goal conflict situation where the person's approach versus avoidance *potentials* are functions of the distance from each goal (which in this case lie in two-dimensional space).

this term is borrowed from physics; let us take a moment to establish the connections.

A good starting point is to think of a goal attracting a person in analogy with a massive body attracting another body of negligible mass. In this case we might try to model the attraction in terms of a single number that applies at a given point in the space (i.e., for a given distance between them and with given coordinates). Because this number will vary over the space, it yields a so-called scalar (i.e., specified by a single number) field. For instance, if the space in which the person and goal, or two objects lie, is a plane, then the scalar field is a two-dimensional surface. An example is shown in Figure 5.3. Under certain conditions, this surface is a so-called "potential," and the partial derivatives with regard to the coordinates are the components of force of attraction, with regard to the x and y axes. Thus, from a single function, the potential, one can immediately obtain the relevant force quantities. Let P = potential, then the partial derivative of P with respect to x or y respectively gives the forces along those coordinates. Further, we may gather these components into a vector, which we call the "gradient," $G = \text{grad } P = (\frac{\partial P}{\partial x}, \frac{\partial P}{\partial y})$.

It then follows from the differential calculus that a small change in the potential as we move a small distance in the space, is given us by the dot product (i.e., inner product) of this vector and the vector of change of position in the space:
$dP = G.dr$, where $dr = (dx,dy)$, the vector of changes in x and y. That is,

$$dP = \frac{\partial P}{\partial x} dx + \frac{\partial P}{\partial y} dy.$$

In a single dimension, all that was ever considered before, the gradient is just the derivative of the potential and is also therefore equivalent to the force itself. The potential and the gradient have many uses in applied mathematics, even outside physics, especially with regard to optimization and asymptotic dynamic behavior. This is largely because the direction of the gradient vector is that of maximal change of the field or surface for any point on that field.

At this point we return to Miller's postulates for more mathematical interpretation. Postulates 1 and 2 imply that the slopes or derivatives of the gradients should be positive in the direction of the goal, and 3 means that the avoidance gradient increases faster than the approach gradient near the goal so that the aversive qualities become stronger, faster as one approaches the goal. Postulate 4 asserts that the entire gradient must rise monotonically with positive (e.g., hunger, power, sex) or negative (e.g., fear, hate, disgust) drives respectively. A special case would be to multiply the gradient by a positive number (> 1) for each increase in drive. Postulate 5 simply states that the gradients are affected by learning. Postulate 6 says that we may simply subtract the gradients at any point in time or space to learn what the person will do. In a multidimensional vector space, we can still subtract vectors. However, a point that is overlooked in Postulate 6 is that under the traditional interpretation of "gradient," it is, as mentioned earlier, defined by forces that are in turn defined by a second-order differential equation (i.e., it is proportional to acceleration, rather than velocity). Thus, even though the gradients may be equal at a particular point in space giving a resultant force of zero, the velocity may not be zero so in fact the person is still acting or moving in the real or psychological space. This will be illustrated mathematically later. Nevertheless, for many purposes, and especially in spaces with dimension greater than 1, it will suffice to employ first-order systems. This will be made clear in the following.

It may be helpful to review some basics of differential equations to start things off. Such a review is located in the Appendix.

LINEAR APPROACH-AVOIDANCE IN ONE DIMENSION

There is considerable value in working out the situation for a linear gradient in one dimension. For one thing, as noted earlier, Miller (1959) used it, at least in pictorial illustration. The linear case is readily understood and solvable, even in the second order case. It is useful pedagogically, particularly because we learn through it that linearity may always be natural for the approach-avoidance problem in general.

We first inspect the single-goal approach–avoidance situation where a person is both attracted and repulsed by a single goal or choice object. Thus, her/his choice is basically whether the positive aspects of the goal outweigh the negative aspects. However, the approach–avoidance theme immediately places us in the context of a continuum of response possibilities (e.g., how far toward the object shall the person go), rather than simply an all-or-none choice. We feel that this continuum of possibilities or tendencies is really more descriptive of a person's psychology than the more traditional all-or-none depiction. After that, we observe the basic linear dynamics in a two goal-object situation.

First we outline the mathematical situation which involves only a single goal-object. The following equations show the approach and avoidance gradients respectively. Let

$P(t)$ = Position of person at time t.
G = Position of Goal
a_0 = Value of positive approach gradient when $P=G$ (i.e., force when goal is reached)
a_1 = Slope of approach gradient
b_0 = Value of avoidance gradient when $P=G$.
b_1 = Slope of avoidance gradient
k_2 = Slow-down or effort coefficient

In order to best capture the spirit of the theories of Lewin, Hull, and Miller, all the above parameters, including G, should be nonnegative. $P(t)$, of course, can be positive or negative. Then the germane dynamic equations are

$$F^{+}(t) = \frac{d^2P(t)}{dt^2} = a_0 - a_1[G-P(t)] - \frac{k_2}{2}\frac{dP(t)}{dt}$$

$$F^{-}(t) = \frac{d^2P(t)}{dt^2} = -[b_0 - b_1[G-P(t)] + \frac{k_2}{2}\frac{dP(t)}{dt}]$$

Where F^{+} and F^{-} are the forces (gradients) that would determine behavior if only the positive or negative influences respectively were present. Let us first analyze $[a_0 - a_1(G-P(t))]$ for F^{+}, the term $-[b_0 - b_1(G-P(t))]$ for F^{-} is comparable. Already we come upon some awkwardness of the linear approach. Note that the intercept, when $P(t)=0$, is $a_0 - a_1G$ so it has to be a function of the coefficient a_0 as well as the position of the goal G. This is required so that F^{+} can increase as P approaches G as stipulated in the postulates. The coefficient a_1 gives the slope of ascent. So if $a_0 - a_1G > 0$ as we ordinarily assume and if $P(0)=0$ and $\frac{dP(0)}{dt}=0$ indicating that the starting position and velocity are both zero, then the initial force and movement are toward G (neglecting F^{-} for the moment) and the positive force increases as P approaches G. When $P(t) > G$ as will eventually occur, F^{+} becomes even larger and goes off to infinity. This is absurd, needless to say, so we must build in a threshold terminator or a much more complex mechanism so that the person stops on reaching the goal. (Thus, a nonlinearity intrudes in spite of the initial linearity.) But, such mechanisms are very common in the application of differential equations. We shall neglect the expression for that *goal-reached* stop function in this presentation.

However, we do want to include a term for slow-down (deceleration) that might be due to mental or physical effort and the pertinent one is $\frac{-k_2}{2}\frac{dP(t)}{dt}$, the velocity. For simplicity we assume it operates the same way in F^+ and F^-. Observe further that it opposes the direction of movement whatever it is when $k_2 > 0$. (That is, $\frac{-k_2}{2}\frac{dP(t)}{dt} < 0$ if velocity is forward and vice versa if $\frac{dP(t)}{dt} < 0$

The typical drawings of the gradients, as in Fig. 5.4 $\frac{dP(t)}{dt}$ for which we would require an additional coordinate. Because we assume $\frac{dP(t)}{dt} = 0$ at $t = 0$ the early behavior is determined primarily by $P(t)$, so this does not do any real harm. The major effect of $\frac{-k_2}{2}\frac{dP(t)}{dt}$, from our point of view is on behavior around a point of equality, when $F^+(t) = F^-(t)$ as we shall see shortly.

Note that the equations are in the second degree so that we are effectively talking about force (without the precisely defined units to which physics is privileged). Figure 5.4 schematizes the gradients. We do not lose any generality by assuming that the person starts at the origin as noted and moves toward or away from the goal, G. There are good a priori reasons as well as experimental evidence that the avoidance gradient should ordinarily be steeper than the approach gradient as postulated, ($a_1 < b_1$, see following discussion) but there might be pathological cases where this would not be true. The point of intersection of the two gradients will also depend on the environmental and psychological circumstances. The overall resultant force is determined by the sum of the positive and negative gradients, that is,

$$F = F^+ + F^- = (a_0 - b_0) - (b_1 - a_1)P(t) - (a_1 - b_1)G = k_0 - k_1 P(t);$$

$$k_0 = a_0 - b_0 - (a_1 - b_1)G;\ k_1 = b_1 - a_1$$

Obviously postulates 1, 2, and 3 are easily captured in our model. Postulates 4 and 5 are implemented by assuming that the intercept and or slope parameters a_0, a_1, and b_0, b_1 are increasing functions of drive and learning. Miller gives no guidance as to whether the intercept or slope or both should be affected by either drive or learning. If we look to Hull, drive and learning should multiply the entire gradient equation and therefore affect both. In any event it is clear we can embed sufficient structure in our model to incorporate all the postulates.

Now, assume $k_1 > 0$ and let $k_2 = 0$. That is, the avoidance slope is sharper

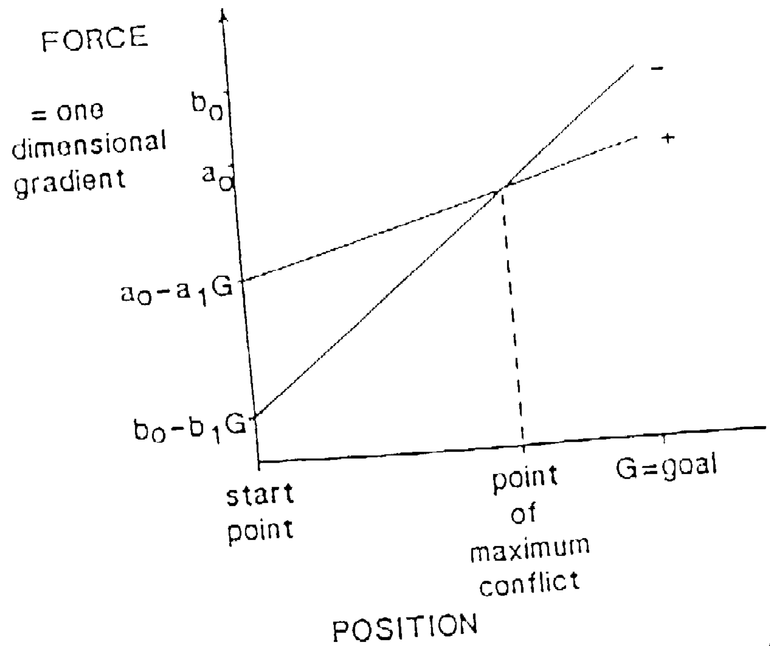

FIG. 5.4. A typical single-goal approach–avoidance conflict situation where the person is both attracted to and repelled by a single-goal. The (linear) approach (+) and avoidance (−) gradients show, respectively, the forces that would apply if only positive or negative influences were present. See text for further discussion.

than that for approach and there is no slow-down due to effort. Using standard techniques, we find that the general solution to this case (e.g., see Luenberger, 1979) is

$$P(t) = \beta_1 \cos wt + \beta_2 \sin wt + \frac{k_0}{k_1}$$ where β_1 and β_2 must be

determined by the initial conditions and $w = \sqrt{k_1}$. Suppose that the initial velocity (i.e., the first derivative) is zero. Then the pertinent solution is

$$P(t) = \{\frac{k_0}{k_1}\} (1\text{-}\cos wt)$$

and we see that the person oscillates around the point of maximal conflict, that is, where the two gradients are equal. The frequency of oscillation is given by $\sqrt{k_1}$ and the amplitude by

$$\left|(k_0/k_1)\right| = \left|\frac{(a_0 - b_0) - (a_1 - b_1)G}{b_1 \quad - \quad a_1}\right|$$

If the amplitude exceeds the distance from the crossover point to the goal, then we would expect the person to absorb (i.e., choose, etc.) at the goal, presumably getting both the goodies as well as the baddies associated with the goal. Figure 5.5 illustrates the waveform of oscillation and below that, the point of oscillation, in the dimension of activity.

Now let $k_2 \neq 0$. The most intuitive version is with $k_2 > 0$, that is, a cost associated with "movement." We find that if $0 < k_2 < \sqrt{4k_1} = 2\sqrt{b_1 - a_1}$ then oscillation still results but the magnitude of this oscillation decreases to zero and ultimately the person goes to a point (k_0/k_1) at which $F^+ = F^-$ and stops. The solution in the latter case is given by

$$P(t) = \frac{k_0}{k_1} + c_0\, e^{\lambda_0 t} + c_1\, e^{\lambda_1 t}$$

where k_0 and k_1 are as above, and

$$\lambda_0 = \frac{-k_2 + \sqrt{k_2^2 - 4k_1}}{9}$$

$$\lambda_0 = \frac{-k_2 - \sqrt{k_2^2 - 4k_1}}{9}$$

Actually, the oscillations around (k_0/k_1) converge to zero in magnitude as t approaches infinity. Note that for $0 < (k_0/k_1) < G$, the point of equilibrium (k_0/k_1) lies between the start position and the goal.

The unusual case where the approach gradient is steeper than the avoidance gradient can be studied in the aforementioned model also. Miller (1959, p. 222) discussed this possibility. In addition to material not directly related to

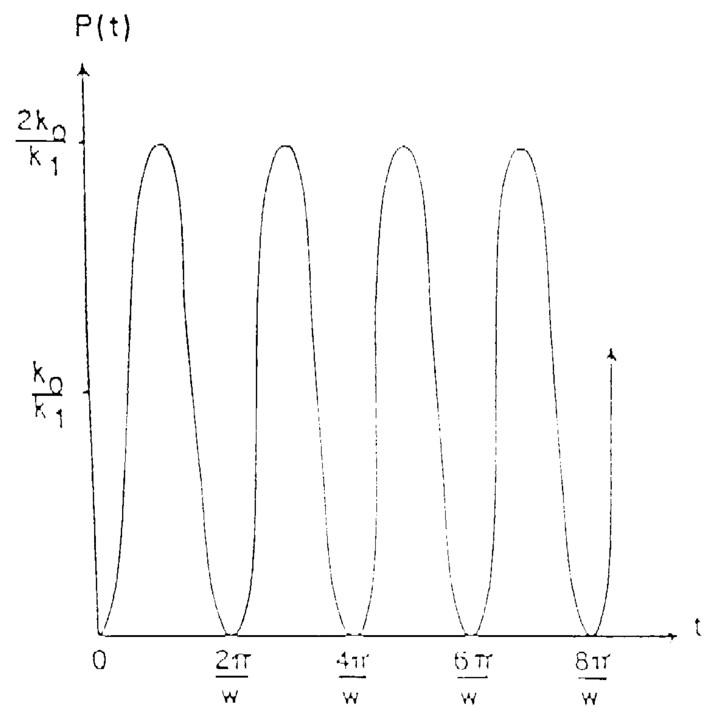

FIG. 5.5. The waveform of a person oscillating, over time, in space (psychological or physical) around the equilibrium point k_0/k_1

approach–avoidance, Miller went on to discuss matters of displacement of aggression and stimulus generalization. These are interesting and will be pursued elsewhere. Here we consider generalizations to two-goal situations and then some elementary approach–avoidance dynamics in higher dimension.

LINEAR DOUBLE APPROACH–AVOIDANCE IN ONE DIMENSION

Suppose a person faces the dilemma of two choice objects, each with its own positive and negative attributes. What does he/she do? This environment can be pictured as in Fig. 5.6 where the person starts at the origin again and on the right lies Goal 1 and on the left Goal 2. The prototypical case is shown there with the avoidance gradients being steeper than those for approach and with the approach gradient starting higher. Again ignoring the slow-down term governed by k_2 for simplicity, the two sets of resultant dynamic equations pertinent to the two goals G_1 and G_2 are

$$F_{G_1} = \frac{d^2P_1}{dt^2} = k_0 - k_1\, P(t)$$

$$\text{where } k_0 = a_0 - b_0 + (b_1 - a_1)G_1$$
$$k_1 = b_1 - a_1$$

and

$$F_{G_2} = \frac{d^2P_2}{dt^2} = l_0 - l_1\, P(t)$$

$$\text{where } l_0 = c_0 - d_0 + (d_1 - c_1)G_2$$
$$l_1 = d_1 - c_1$$

and the parameters in F_{G_2} function exactly analogously to their counterparts in F_{G_1}, but with respect to G_2, to the left of the start position. Overall then,

$$F = F_{G_1} + F_{G_2}$$
$$= (k_0 - l_0) - (k_1 - l_1)P(t)$$

and the several solutions are obtained in the same way as before. In particular the person will go to G_1, go to G_2 or end up at a point of positive and negative gradient equality in the event that the point of maximum conflict lies between G_1 and G_2, and if the excursion of any oscillation does not hit G_1 or G_2.

Thus, the range of qualitative behavior is like the single approach–avoidance case but the actual movements now depend on two rather than one goal and the related attractive and repulsive aspects of those two goals.

We next move to the two dimensional choice domain, that is, when the

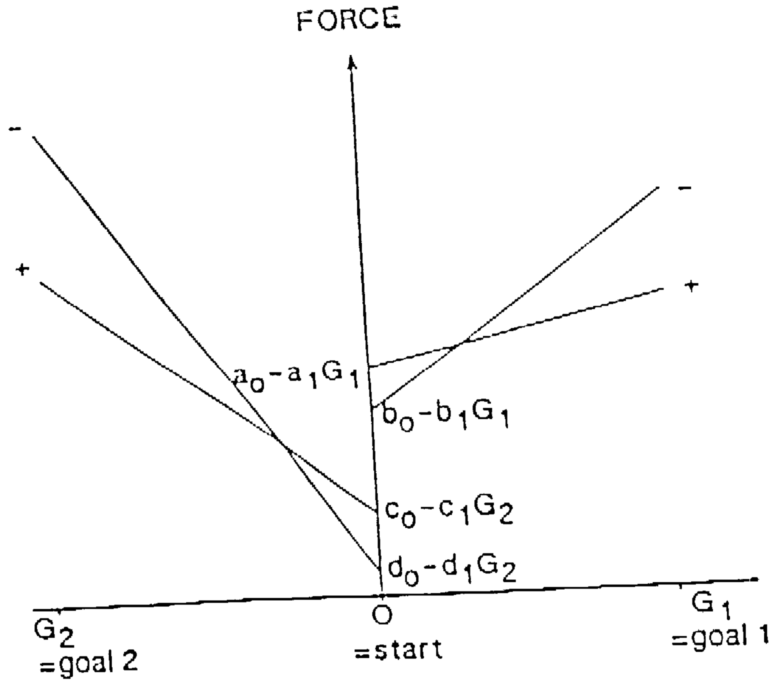

FIG. 5.6. An example of approach (+) and avoidance (−) gradients in a double-goal situation where each has both positive and negative attributes. As in the single-goal situation, the avoidance gradients are steeper than the approach gradients, and the approach gradients start higher.

choice space is the plane. At this point, the linear model becomes rather cumbrous so we elect to adopt a more general approach and then discuss some nonlinear special cases of interest.

MULTIDIMENSIONAL APPROACH-AVOIDANCE BEHAVIOR

We shall stay within the context of two dimensions, but the formalisms can clearly be immediately generalized to arbitrary finite dimensions. We require more structure now and therefore introduce the following new set of assumptions.

Assumptions for Multidimensional Approach-Avoidance

1. The vector representing approach toward a single goal should point directly toward that goal and a vector representing avoidance of a single goal should point directly away from that goal.
2. We define a potential that is a decreasing function of distance from a goal, $p(d)$.
3. We then extract a gradient from that potential by taking the partial derivatives of the potential with respect to the two dimensions, x and y; that is $\frac{\partial P}{}$ and $\frac{\partial P}{}$

4. The gradient will then be a vector whose components represent the tendency toward or away from the goal on the two dimensions; that is $g(x,y) = (\frac{\partial P}{\partial x}, \frac{\partial P}{\partial y})$.

5. The length of the gradient vector gives the overall strength of approach or avoidance from the present location, P; that is if V = Valence or strength,

$$V = \sqrt{(\frac{\partial p}{\partial x})^2 + (\frac{\partial p}{\partial y})^2}$$

6.* The length (magnitude) of the gradient vector is a decreasing function of distance from the goal.

7. "Approach" in a given environment and person is defined by a family of gradients that point toward a goal and "Avoidance" is another family of functions that point away from the goal. To avoid triviality, we assume that motivational circumstances range from the case where approach always dominates, where avoidance always dominates, and where they cross. Where they cross corresponds as usual to a point of maximum conflict. The total gradient is given by the sum of the constituent gradients.

In most circumstances these assumptions will be sufficient to guarantee that there exists a point of equality of approach and avoidance for a single goal, and along a single line connecting the present location to the goal, and without consideration of other goals, that such a point be unique. This usually follows from the fact that the numerical difference of the approach and avoidance gradients will be an increasing function of distance. This function would go from a negative quantity through zero (the point of maximum conflict) on to positive quantities as distance increases from zero to large values.

Assumption 1, besides seeming psychologically reasonable, avoids messy mathematical expressions and certain complicated and mystifying behavior. For instance, cases might arise where approach equals avoidance on the two dimensions at different distances, which generally precludes a cross-over of the gradients and thus a point of maximum conflict.

The second assumption links up the present development with classical physics and says how the potential, which stands for a sort of primitive striving (akin to m/d, the attractive potential between two objects in physics, where m = product of the two masses and d = distance) toward or away from a goal. The following assumption states that we can find the components of striving on each dimension by taking the derivative of the potential with regard to each dimension. The next, number 4, simply puts the result into vector form and defines it as the gradient. The fifth assumption defines an overall valence as the magnitude (or length) of the gradient vector whereas the sixth requires that it, too, be a decreasing function of distance. The latter condition is not implied by Assumption 2, as can be seen by its expression.

Assumption 6 is starred(*) because it seems more optional. In a sense, Assumption 6 demands that the actual "movement" increase in velocity as one approaches the goal. To the extent that motor control in physical motion is ignored, this might make sense (and occurred in the one-dimensional pure approach situations discussed earlier in the linear systems examples). Otherwise, it is acceptable to permit the velocity (or force) to go to zero as one gets very close to the goal, as would happen in a smooth dynamical system acting in real time and space. Note that the more primitive notion of a gradient that increases as one approaches the goal, can be preserved even when the organism slows down in approaching it.

Interestingly, it turns out to be a little tricky to come up with a class of functions obeying all seven assumptions. Let the goal G be at the origin $G=0$. Later examples flout Assumption 6, but a two-dimensional family of functions that *will* satisfy all seven is given by the system of differential equations,

$$\frac{dx}{dt} = \frac{-axe^{b(x^2+y^2)^{-1}}}{(x^2+y^2)^2},$$

$$\frac{dy}{dt} = \frac{-aye^{b(x^2+y^2)^{-1}}}{(x^2+y^2)^2}, \text{ where } b > 0.$$

Composing this system into a velocity vector we can then write

$$g(x,y) = \frac{-a\, e^{b(x^2+y^2)^{-1}}}{(x^2+y^2)^2}(x,y) = F(D)\,(x,y), \text{ with } F(D) \text{ a decreasing function}$$

of distance D and (x,y) the current position vector. The vector $g(x,y)$:

1. Points directly toward ($a > 0$) or away ($a < 0$) from the goal $G=(0,0)$.
2. Comes from a potential $p(D) = \frac{a}{2b}\exp(b/D)$—where D equals x^2+y^2,

that is, the euclidean distance squared, from the goal. This function $p(D)$ is a decreasing function of distance from the goal.

3. If we write a single approach–avoidance conflict situation as

$$g_{ap}(x,y) - g_{av}(x,y) = \left\{ -\frac{a\, e^{b(x^2+y^2)^{-1}}}{(x^2+y^2)^2} + \frac{ce^{d(x^2+y^2)^{-1}}}{(x^2+y^2)^2} \right\}(x,y)$$

where $c > a$ and $d > b$, then there exists a unique set of points of maximum conflict where approach tendency equals avoidance tendency, approach dominates beyond that point and avoidance dominates when the organism is closer to the goal than that point. These points all satisfy $x^2+y^2 = -\{1/(d-b)\} \cdot \log(a/c)$.

4. The magnitude of the gradient as well is a decreasing function of distance from the goal.

We next use a less complex gradient that satisfies all the stipulations except number 6 and in several special cases, even that one. Consider the potential $p = a/(x^2 + y^2 + b)$. By differentiation, we then get the gradient $(dx/dt, dy/dt) = (-2ax/(x^2 + y^2 + b)^2, -2ay/(x^2 + y^2 + b)^2)$ where we can immediately absorb the "2" into the constant "a". The magnitude (length) of gradient vector is not monotonic decreasing in $\sqrt{D} = \sqrt{x^2 + y^2}$ unless $b = 0$, violating Assumption 6, but we need the b when approach–avoidance conflict is present in order to produce a cross-over of the positive and negative gradients. A separate potential is used to construct approach and avoidance gradients. The approach and avoidance gradients are added to produce the net gradient

$$\left(\frac{-ax}{(x^2 + y^2 + b)^2} + \frac{cx}{(x^2 + y^2 + d)^2} \cdot \frac{-ay}{(x^2 + y^2 + b)^2} + \frac{cy}{(x^2 + y^2 + d)^2}\right).$$

The first example is of a single goal approach–avoidance situation. As in the other two-dimensional models, we let the single goal be placed at the origin $G = (0,0)$. Figure 5.7a shows the way in which the gradient vectors point at different locations of the display. (The longer arrows will be discussed later.) It can be seen that they all point toward or away from the goal (0,0) as they should. Figure 5.7a is known as a direction field because it gives the direction of the gradients but not their magnitude. If that were also shown, in the present case, they would increase in length as the origin is approached from far away in the display but peak and then drop toward zero length close to the goal. This nonmonotonicity is because of violation of Assumption 6; if that were satisfied, then the vectors would increase in magnitude all the way to the goal.

Also shown in Fig. 5.7a are several "phase portraits" (plots) indicating the paths taken from various starting points. "Phase portrait" is simply a conventional name of a picture of the trajectory (or path) that the dynamic body (in our case, a person) takes through the space. The word "phase" has no particular relevance for us in the present context. Note that the trajectories line up with sets of direction vectors and that time is implicit and invisible. Also note that a set of equilibrium points surrounds the goal at points of maximum conflict. Close to the goal, avoidance dominates and the organism moves toward the circle of equilibrium points. On the other side of the circle, approach dominates and the organism is predicted to move toward it. Because velocity is continuous and goes to zero at the set of equilibrium points, that set is never actually met in finite time, only approached. Such an approach is depicted in Fig. 5.7b where we watch coordinate x approach the particular equilibrium point to which it is attracted, as a function of time and from two approach and two avoidance starting values.

We next go to a double goal approach–avoidance environment. Here it suffices to use the two goals $G1 = (0,0)$ and $G2 = (A,0)$. No loss of generality is entailed by letting both y goal components be 0. We may illustrate the dynamics with a model also obeying Assumption 6 by setting the b-parameter

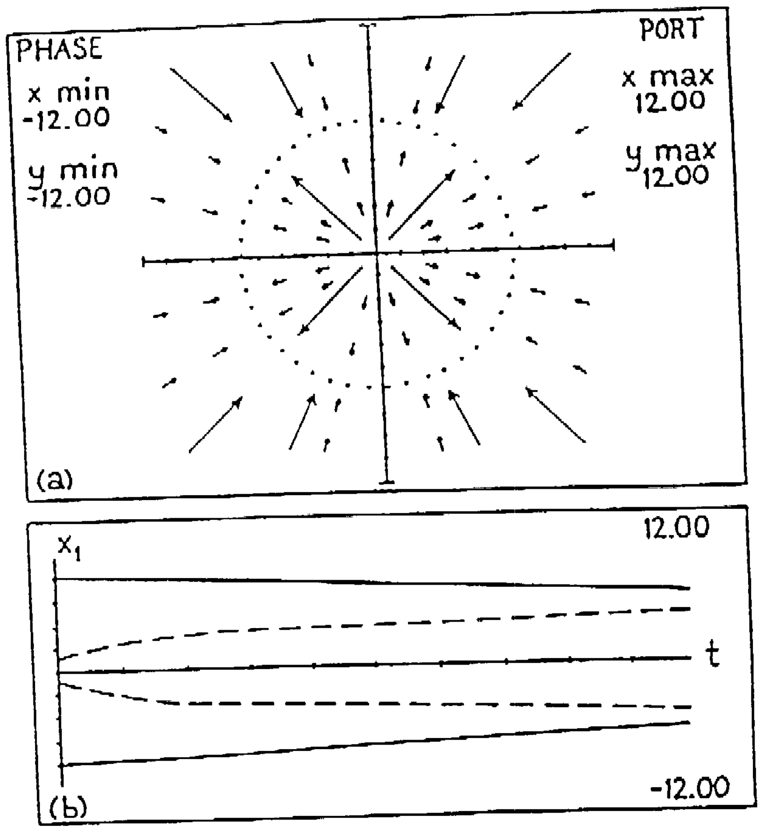

FIG. 5.7. (a) Direction field and several phase portraits, corresponding to different starting points, of a single goal approach–avoidance situation. The dotted circle represents a set of equilibrium points. (b) A plot of the position x_1 as a function of time. The solid lines are examples of starting points greater than the equilibrium points and thus x_1 is "approaching" the goal, whereas the dashed lines are for starting points close to the goal—inside the circle—so these are "avoiding" the goal.

equal to zero. Also let $a = 1$ for simplicity. Now the gradient of approach toward $G1$ is $g_1(x,y) = \left(\frac{-x}{(x^2+y^2)^2}, \frac{-y}{(x^2+y^2)^2}\right)$, and that of approach toward $G2$ is $g_2(x,y) = \left(\frac{-(x-A)}{((x-A)^2+y^2)^2}, \frac{-y}{((x-A)^2+y^2)^2}\right)$. The overall gradient is found by adding the two:

$$g = g_1 + g_2 = \left(\frac{-x}{(x^2+y^2)^2} - \frac{(x-A)}{((x-A)^2+y^2)^2}, \frac{-y}{(x^2+y^2)^2} + \frac{-y}{((x-A)^2+y^2)^2}\right).$$

The gradient vector field appears in Fig. 5.8 and now it can be seen that the vectors often point in directions intermediate between the two goals. However, the closer the organism is to one of the goals, or if one lies entirely to the right or left of both goals, the more directly the vector points toward the nearest one. Several phase plots are given in Fig. 5.8 and the geometric fact that the direction field vectors are tangent to a phase portrait solution becomes obvious: (Fig. 5.7a is a special case where any solution is a straigh

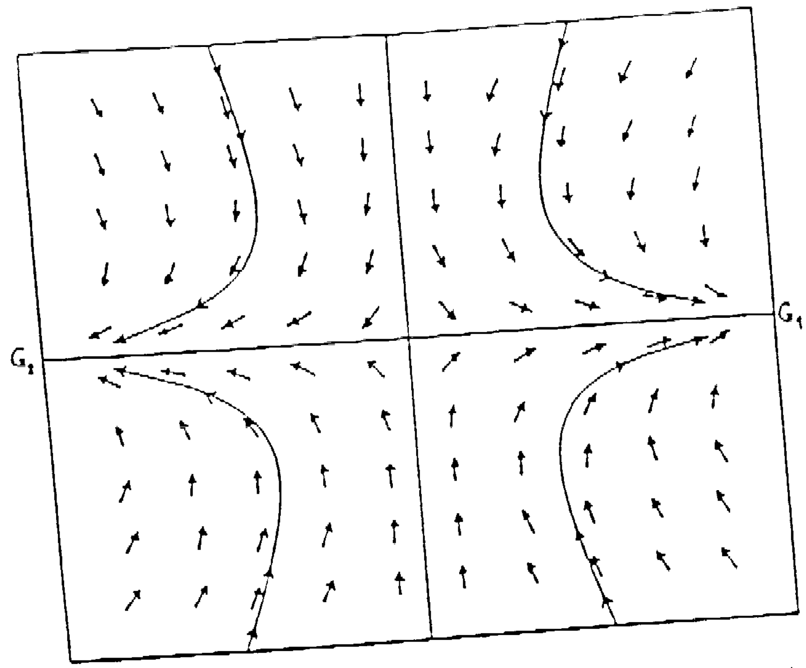

FIG. 5.8. Gradient vector field (short arrows) and several phase plots (long arrows) approach–approach situation.

line so that all tangents lie exactly on it). Yet another feature of Fig. 5.8 is the line through the graph, separating the qualitative types of activity; it is known for this reason as a "separatrix." That line happens to coincide with the y-axis with this illustration, but in general would not. If a person were to start exactly on this line, he would never move, but rather like a pendulum balanced at the top of its swing over its pivot, any slight perturbation would cause immediate movement toward the strongest attracting goal.

The final example is a double avoidance–avoidance situation in which both goals are aversive, with no redeeming aspects. The qualitative theory of Lewin suggests that given the chance, the person will attempt to escape the area close to the aversive objects mentally or physically, as the case may be. We shall see that exactly this is predicted by our mathematical dynamics. The appropriate gradient is given by the equations for the approach–approach case but with the signs of the vector components all reversed. Figure 5.9 illustrates the direction field and some phase plots.

Observe that the vectors generally point away from the negative goals and suggest that a person will leave the field proximal to these goals. The phase portrait paths superimposed on the direction field confirm this suspicion. It is interesting that when the person is on the line joining $G1$ and $G2$, she cannot escape and must either come to rest at the best possible distance from both (as in the present case) or oscillate forever between them. This set of points is unstable, for any tiny displacement off of this line will propel the organism out of the immediate field.

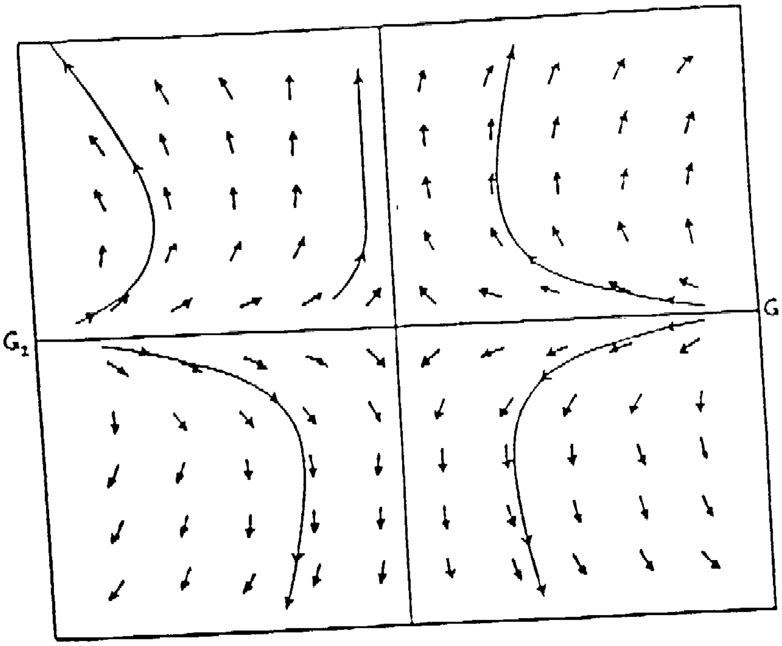

FIG. 5.9. Gradient vector field and several phase plots of a double-goal avoidance–avoidance conflict situation.

Related Theories

Two theories that use dynamic models to describe decision and choice are Atkinson and Birch's (1970) "dynamics of action" theory and Grossberg's (1978, 1980) competive systems theory. Both theories describe the dynamics of choice behavior by systems of differential equations. What is unique about the present work is the inclusion of the goal gradient hypothesis into the dynamics—the idea that intensity of motivation is a function of the distance from the goal. Although this idea could be built into the previous two models through attentional mechanisms or other means, this was not explicitly considered, and the present chapter appears to provide the first formal treatment.

Aside from the goal gradient structure the present theory does not fit readily into the competitive system framework (Grossberg, 1978, 1980). One reason is that the latter views each unit as an entity that competes through dynamic coupling with the other unit. This produces a system of differential equations in the various units. In our system, by way of contrast, the gradients sum for an arbitrary position on the part of a single "unit." This results in a single differential equation driving the dynamic behavior of the "Unit" (read person or decision mechanism). Atkinson and Birch (1970) also treat their "action tendencies" as separate entities that get integrated before comparison.

Nevertheless, one way to try to place an approach like ours into such a competitive system framework might be the following. Consider the double goal problem on a line, with a starting position at $P(0)=0$, and the goals located at $-G$ and G. The attractiveness of each alternative at any point in time can be represented by the variables $x_1(t) = P(t)/G$ and $x_2(t) = -P(t)/G$, where $P(t)$ is the position at time t. According to Grossberg (1980, p. 382) this two alternative system is a competitive system if $\frac{\partial}{\partial x_2}\frac{dx_1}{dt} < 0$, and $\frac{\partial}{\partial x_1}\frac{dx_2}{dt} < 0$. According to our one dimensional double goal model $F = k_0 - k_1 \bullet P(t)$ and $k_1 > 0$. If it is assumed that $F = \frac{dP(t)}{dt} = \frac{G\ dx_1(t)}{dt}$, then $\frac{\partial}{\partial x_2}\frac{d}{dx} x_1\ (t) = k > 0$, which fails to satisfy the competitive property.

More recently, Grossberg and Gutowski (1987) proposed a rather different dynamic model of risky decision making called affective balance theory. There are two major differences between our approach–avoidance theory and affective balance theory. First the latter does not incorporate the goal gradient hypothesis. Second, according to approach–avoidance theory, the competing forces produced by each alternative are processed in parallel and combined at each moment in time, but according to affective balance theory, the alternatives are processed sequentially. The forces for the first alternative are integrated over time, and the integrated value is compared to the value of the second alternative. Among other consequences this leads to different predictions concerning decision time.

WHITHER-TO NEXT?

There are many directions worth following up along the lines initiated earlier. Some of these are listed here.

1. Most experiments directly pertinent to the tenets of this theory have been done with rats. It is important to expand the studies to the human domain and specifically to be able to use negatively valenced goals that are in truth aversive to the subject, yet are harmless.

2. Among a number of particular domains of application in the social sciences, a natural and compelling one is risky decision making; that is, decision making when outcomes are uncertain and may be associated with various consequences. This has been a region of high priority in utility theory. We suggest that dynamic approach–avoidance theory can make a contribution here (cf. Townsend & Busemeyer, 1987).

3. In addition to refining and testing the theory in more numerically oriented ways with new data, it is of interest to generalize the ideas with mathematical structure not limited to euclidean vector spaces or even orthogonal coordinate spaces. Thus, investigations into manifold theory would seem appropriate (e.g., see Irwin, 1980) as would topological dynamics based on continuity and the notion of a metric but sans differentiability (e.g., see Sibirsky,

1975). Possibly less dramatic but of high importance would be the implementation of the qualitative theory of differential equations, particularly general notions of stability and limit cycles.

4. Related to (3) yet critical in its own right, is the investigation of probabilistic (read stochastic) versions of the theory. Although this must be undertaken in the near future, we believe that the deterministic theory should be developed first or at least in parallel with the stochastic theory. We suspect that something deterministic lies at the bottom of the behavior, a skeleton as it were, that is fleshed out by stochastic properties. Different versions of the stochastics can yield vastly different types of behavior, though the underlying skeleton is the same. Psychologists rarely have much to go on in deciding what stochastic structure should apply in any given milieu. Alternatively, the stochastic appearance of much human behavior could be due to deterministic, but chaotic underpinnings (see, e.g., Devaney, 1986).

5. It may be that approach–avoidance theory has something to offer practical decision making theorizing. Although utility theory and statistical decision making can represent negative vs positive gains, it may be that the emotional and motivational aspects of many, if not most, important decisions can best be captured by our type of theory. Certainly the time-dynamic character would seem valuable in accurately describing real-life decision behavior. Close in conception would be the application of these ideas in robotology and artificial intelligence, especially in situations where a mimicking of human motivational properties is desired. This in turn could help to link up goal-directed behavior in computers or robots with the burgeoning field of neural modeling and neural computation.

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APPENDIX
ASPECTS OF DIFFERENTIAL EQUATIONS

Order of a Differential Equation

A differential equation describes the dynamics of a body (object, idea, etc.) by way of its derivatives. We may metaphorically think, in our case, of the first derivative as being "like" a velocity, the second as being like an acceleration and so-on. The "order" of a differential equation is the highest derivative appearing in the equation. An example of a second-order differential equation is

$$\frac{d^2x}{dt^2} = (\sin x)\, t^2 = +2 \qquad (1)$$

The independent variable is time $= t$ and the dependent variable is x, usually a position in some one-dimensional space (e.g., taken with respect to a goal in our situation).

Linear versus Nonlinear Differential Equations

A "linear differential equation" is one where all of the terms are linear combinations of the dependent variable and its derivatives, possibly plus a term not involving the dependent variable. The coefficients of these dependent variable terms can in general be functions of the independent variable, as can the extra term that does not involve the dependent variable. An example of a second order linear differential equation is

$$t^3 \frac{d^2x}{dt^2} = t\,\frac{3dx}{dt} + \text{Log } t + 1 = 0. \qquad (2)$$

Note that it is okay for the equation to possess nonlinear functions of the independent variable. It follows from the aforementioned that a nonlinear differential equation must have at least one nonlinear function of the dependent variable or one or more of its derivatives. Two examples are the following:

$$\left\{\frac{d^2x}{dt^2}\right\}^2 = 5\sqrt{t} \qquad (3a)$$

$$6\frac{dx}{dt} - Log(x) = 0 \qquad (3b)$$

In the first, the second derivative is squared, a nonlinear operation and in the second, one finds Log(x), also a nonlinear operation. Equation 1 from earlier, is also nonlinear because sin(x) is nonlinear in x.

Homogeneous versus Nonhomogeneous Differential Equations

A "homogeneous differential equation" is one where terms not involving the dependent variable are absent. Equivalently, terms involving only the independent variable or constants are lacking. If it is present, then we have a "nonhomogeneous differential equation." This extra term is usually thought of as a forcing function that serves as an input to the system; that is, it "drives" the system. Thus, in contrast, a homogeneous differential equation describes a system whose entire behavior from a particular point in time is determined by where it starts and the initial values of certain of its derivatives. A nonhomogeneous system must in general include influences both from the forcing function as well as initial values. An example of a homogeneous differential equation is that of Equation 3b. Equation 1 has a forcing function given by the number "2" and Equation 3a by $5\sqrt{t}$ so both of these are nonhomogeneous. (How about Equation 2?).

The behavior of a linear system (i.e., a system defined by linear differential equations) can always be determined by adding the solution for the general homogeneous equation to a particular solution obtained with the forcing function present, but with all pertinent initial values set to zero.

Transient versus Asymptotic Behavior

We can also ask about the short term behavior of a system as well as how it acts as the duration under observation increases without limit. We may point out that while an elegant general theory of linear systems exists that perfectly describes both transient and asymptotic behavior, the only reasonably general theory that applies to large classes of nonlinear systems pertains to asymptotic behavior. Further, suitably defined "stable" linear systems have the property that the equation describing their behavior can be decomposed into a set of transient terms (i.e., terms that die out, that is go to zero) plus a set of asymptotic terms (i.e., give the position to which the system converges).

Systems of Differential Equations

When there is more than a single output of a system, or it is operating in a multidimensinoal space, as in our more general cases, a set of differential equations is needed. For instance, in the plane, we would ordinarily see a system of two differential equations, which may be generally written as

$$\frac{d^2x}{dt^2} = f(x,y,\frac{dx}{dt},\frac{dy}{dt},t),\ \frac{d^2y}{dt^2} = g(x,y,\frac{dx}{dt},\frac{dy}{dt},t) \qquad (4)$$

when we have a second order system.

A general first order two-dimensional system would be written,

$$\frac{dx}{dt} = f(x,y,t),\ \frac{dy}{dt} = g(x,y,t) \qquad (5)$$

Observe that f and g may in general be nonlinear and nonhomogeneous.

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