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Serial and Within-Stage Independent Parallel Model Equivalence on the Minimum Completion Time[1]

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A set of functional equations is investigated that, when satisfied, yields equivalence on the minimum completion time between serial models and parallel models having the property of within-stage independence. Within these classes of models, it is shown that any parallel model is equivalent to a serial model, but not any serial model is equivalent to a parallel model. The necessary and sufficient conditions for a serial model to have an equivalent parallel counterpart are observed and then two sufficient conditions on the survivor functions that produce this result are exhibited. A number of examples satisfying the various theorems are discussed and a special case leading to an extension of a property of exponential distributions is derived.

INTRODUCTION

Parallel processing refers to simultaneous processing of two or more elements by a system where an element, for psychological purposes, is typically a construct undergoing some type of sensory, cognitive or motor operations. Serial processing, on the other hand, refers to processing of elements one at a time, and the convention of zero switching time between elements is assumed. In addition to these conditions it is convenient to assume that all elements to be processed are available at $t = 0$ to the system and, that a parallel system begins processing all elements simultaneously, although they need not all be completed simultaneously. An added condition to the definition of serial systems is that an element be completed before the succeeding one is begun.

In stochastic systems, that is, those in which the elements are completed probabilistically in time, the probabilities of different orderings of completion of the available

[1] I am grateful for the opportunity to have spent the academic year 1972–1973 at W. K. Estes' laboratory at Rockefeller University under Grant GM16735, which provided the opportunity for contemplation and continuity of thought, so facilitory to work of a theoretical nature. A good deal of the present work was formulated during that period. I also enjoyed a number of interesting and fruitful discussions with Steven Reder on questions related to this material at that time. W. K. Estes, W. H. Batchelder, and two referees provided helpful comments on exposition.

elements are determined by a set of probabilities on the different permutations of the spatial or temporal positions of the elements in a serial system; and these probabilities may be entirely independent of all other characteristics of the system. In parallel stochastic systems, the different permutations of element completion have probabilities that are related to other aspects of the system, in particular to the overall "rates" (e.g., the reciprocal of the individual object's mean completion time) of the elements.

Our brief and qualitative presentation thus far is a consequence of already established more formal and complete introductions to theorizing and problems of model equivalence in parallel and serial processes (e.g., Townsend, 1972; in press). Although the empirical work on parallel vs serial processing has burgeoned in recent years, the aforementioned papers plus related work by Christie and Luce (1956), Thomas (1969), and Shevell and Atkinson (1974) are practically the only attempts to treat systematically some aspects of the parallel–serial question.

Townsend (1972) gives a functional equation that is necessary and sufficient for parallel–serial equivalence to hold when there exist $n = 2$ elements to be processed. Solution of this equation at the level of the minimum processing time effectively solves the parallel–serial equivalence problem for $n = 2$ as will be seen below. Within the classes of models considered here it will be observed to provide a method of solution to equivalence on the minimum completion time for $n > 2$.

To start with, we assume the existence of appropriate probability densities in the models, the latter being identified with stochastic processes intended as representations of the operations of a processing system. The serial models assume that the successive actual processing times of the elements are disjoint, one element's completion being followed instantaneously by the beginning of the subsequent element. It is also assumed that the elements may be processed in more than one order, different orders occurring over trials.

The parallel models considered in this paper are based on the notion of within-stage independence. By stage is meant the interval between the instants of two successive element completions. Thus, stage 1 is the interval from time $t = 0$ up until the first element is completed. It is easily seen that in a serial model, the length of a stage is equal to the time required for an element to be processed. However, in a parallel model, the actual total completion time of a single element will be equal to the sum of the stage durations up until the element is completed; this is because all elements begin processing simultaneously. Within-stage independence implies and is implied by the property that the likelihood of any element's completion is independent of the completion of any other during any single stage (see also Townsend, in press). The idea then, is that at the beginning of a stage a certain set of elements has been completed at a certain set of times and the probability function on the next element to be finished can depend on these elements and their completion times. However, within-stage independence states that this probability function can be written as the product of the separate probability terms of the uncompleted elements at the stage in question.

Another way of putting it is that the probability function at each successive stage is on the minimum completion time of uncompleted elements, and the variables representing time for each of their elements are independent during that stage. Hence, although the total completion times may be overall nonindependent (e.g., positively or negatively correlated) and the stochastic description of the system modeled may change during a trial depending on the evolving order of completion and the respective times, independence holds within a single stage.

The analysis below employs the notion of intercompletion time, which is simply the duration of a stage; it also corresponds to the concept of interarrival time in the literature on counting processes. Therefore, it is clear that within-stage independence of parallel models means that the parallel density of the intercompletion time within a given *single* stage can be written as the product of the separate densities of the as yet uncompleted objects, although these may in turn be dependent on past completions and their times. It is assumed throughout the following development that the intercompletion times are the most that may be observed during a single trial. These define the distributions on total completion times for individual elements.

For a parallel model to be equivalent to a serial model for two objects a, b it is necessary and sufficient that

$$\begin{array}{ccc} \textit{Serial} & & \textit{Parallel} \\ p f_{a1}(t_{a1}) f_{b2}(t_{b2} \mid t_{a1}) & \equiv & g_{a1}(t_{a1})\, G_{b1}(t_{a1})\, g_{b2}(t_{b2} \mid t_{a1}), \end{array} \tag{1}$$

$$(1-p) f_{b1}(t_{b1}) f_{a2}(t_{a2} \mid t_{b1}) \equiv g_{b1}(t_{b1})\, G_{a1}(t_{b1})\, g_{a2}(t_{a2} \mid t_{b1}) \tag{2}$$

for all $t \geqslant 0$.

The expressions (1), (2) give the joint intercompletion time densities for a, b when a is finished first or second, respectively. Considering first the serial (left) side we note that p is the probability that the (serial) system selects element a first for processing, $f_{a1}(t_{a1})$ is the serial density on processing time for a when it is first, and $f_{b2}(t_{b2} \mid t_{a1})$ is the density for b's processing time starting at $t = t_{a1}$ (and a fortiori the intercompletion time between the completion of a and the completion of b), conditioned on the amount of time a required. The second serial expression is for the opposite order, b processed first, with completely analogous interpretations.

Note that the total time required for processing both a, b is given by $t_{a1} + t_{b2}$ or $t_{b1} + t_{a2}$. Note also that the subscripts $a1$, $b1$, $a2$, $b2$ imply that the associated f's can be different; for example, from distinct families of densities.

On the parallel side, the density $g_{a1}(t_{a1})$ plays the same role as $f_{a1}(t_{a1})$. However, since the stochastic process representing parallel processing must have nonzero probability attached to b finishing during the time interval $(0, t_{a1}]$, the component $G_{b1}(t_{a1})$ gives the probability that b was not completed during that time, when its density was, of course, $g_{b1}(t)$. $G_{b1}(t)$ (or $G_{a1}(t)$) is, of course, one minus the cumulative distribution function, and is sometimes referred to as a survivor function due to its use in renewal theory applications to failure times. Then, after a is completed b is viewed as continuing processing with a (perhaps different from $g_{b1}(t)$) density, $g_{b2}(t_{b2} \mid t_{a1})$, conditional on the time required for processing a. Expression (2) again gives the reverse situation, where b happens to be completed before a. Within-stage independence implies that processing during any one stage k (stage k is the state of the system from the completion of the $(k-1)$th element to the kth element) is independent. This is revealed in (1), (2) by the products $g_{a1}(t_{a1})\, G_{b1}(t_{a1})$.

Expressions (1) and (2) give a complete description of the joint probability space on the intercompletion times of a and b and their order.

In order that (1) and (2) be satisfied, it is necessary and sufficient that separately

$$pf_{a1}(t_{a1}) \equiv g_{a1}(t_{a1})\, G_{b1}(t_{a1}), \tag{3}$$

$$(1-p)\, f_{b1}(t_{b1}) \equiv g_{b1}(t_{b1})\, G_{a1}(t_{b1}), \tag{4}$$

as well as

$$f_{b2}(t_{b2} \mid t_{a1}) \equiv g_{b2}(t_{b2} \mid t_{a1}) \tag{5}$$

and

$$f_{a2}(t_{a2} \mid t_{b1}) \equiv g_{a2}(t_{a2} \mid t_{b1}). \tag{6}$$

This may be seen by holding t_{a1} and t_{b1} at any arbitrary values and integrating over t_{b2} and $t_{a2} \in [0, +\infty)$; this establishes (3) and (4). Then (5) and (6) follow by cancellation of the (3) and (4) terms from (1) and (2).

Equations (5) and (6) may be made to hold by fiat, simply letting the parallel and serial conditional densities at stage 2 be equivalent. There is nothing about serial and parallel processes that this violates and it is compatible with earlier results found in special cases (Townsend, 1972). The interest then, for $n = 2$, resides in (3) and (4). Put another way, the joint probability function on a being first, and this event occurring at time t_{a1}, must be equivalent for parallel and serial processing. It will be seen that this condition is not so trivially satisfied.

Since we can now afford to neglect the second stage we may simplify (3) and (4) slightly to our advantage,

$$pf_a(t) \equiv g_a(t)\, G_b(t), \tag{7}$$

$$(1-p)\, f_b(t) \equiv g_b(t)\, G_a(t). \tag{8}$$

The system of Eq. (7) and (8) is, in a sense, the atom of equivalence between parallel and serial models of the present ilk for $n = 2$. It is as well the condition for equivalence on the minimum completion time between serial models and independent parallel models. Consider a horse race where, in the serial case, horse a is allowed to run with probability p and horse b with probability $1-p$ and the time recorded in either instance. The nonselected horse does not get to run. In the parallel case, as in most horse races, both horses run simultaneously and independently and the winner's time (only) is recorded. When (7) and (8) are satisfied, the implication is that it cannot be determined from the 'winning' time whether both horses were running simultaneously or only one horse was run. It further turns out when $f_a(t) = f_b(t)$ $(t \geqslant 0)$ that solutions to (7) and (8) yield an interesting generalization of the lack of memory of exponential distributions.

The next section discusses necessary and sufficient conditions for solution to (7) and (8) and the form the solutions take when they exist. Examples are also considered here, and in the third section an example is brought out that does not permit (7) and (8) to be solved for the parallel functions in terms of the serial. A fourth section develops the parallel solution when the serial densities are equivalent and the fifth section generalizes the functional equation to arbitrary n. The sixth and final section discusses applications and some related ongoing work.

Solutions to the Functional Equations

THEOREM 1 (Existence). (A) *If g_a, g_b are given, then there exist solutions to* (7) *and* (8) *of the serial terms as functions of the parallel terms and they are of the form*

$$p = \int_0^\infty g_a(t')\, G_b(t')\, dt',$$

$$f_a(t) = (1/p)\, g_a(t)\, G_b(t), \tag{9}$$

$$f_b(t) = (1/(1-p))\, g_b(t)\, G_a(t). \tag{10}$$

(B) *If p, f_a, g_b are given and if there exist solutions to* (7) *and* (8) *of the parallel terms as functions of the serial terms, then they are of the form*

$$G_a(t) = \exp\left[-\int_0^t \frac{pf_a(t')}{pF_a(t') + (1-p)\, F_b(t')}\, dt'\right], \tag{11}$$

$$G_b(t) = \exp\left[-\int_0^t \frac{(1-p)\, f_b(t')}{pF_a(t') + (1-p)\, F_b(t')}\, dt'\right]. \tag{12}$$

Proof of (A). The proof of (A) is immediate. Integration of both sides of (7) (and of (8)) yields the solution of p (and $1-p$). Then f_a, f_b are well-defined densities obtained by dividing both sides of (7) and (8) by this solution for p, $1-p$, respectively, resulting in (9) and (10).

Proof of (B). Add (7) and (8) and integrate from t to $+\infty$. This results in

$$\int_t^\infty [pf_a(t') + (1-p)\, f_b(t')]\, dt' = \int_t^\infty [g_a(t')\, G_b(t') + g_b(t')\, G_a(t')]\, dt'$$
$$= pF_a(t) + (1-p)\, F_b(t) = G_a(t)\, G_b(t). \tag{13}$$

These are both well-defined survivor functions. Now divide the left-hand side of (7) and (8) by the left-hand side of (13) and the right-hand side of (7) and (8) by the right-hand side of (13) to obtain

$$\frac{pf_a(t)}{pF_a(t) + (1-p)\,F_b(t)} \equiv \frac{g_a(t)}{G_a(t)},$$

$$\frac{(1-p)\,f_b(t)}{pF_a(t) + (1-p)\,F_b(t)} \equiv \frac{g_b(t)}{G_b(t)},$$

and integrate both expressions from 0 to t:

$$\int_0^t \left[\frac{pf_a(t')\,dt'}{pF_a(t') + (1-p)\,F_b(t')}\right] \equiv \int_0^t \left(\frac{g_a(t')}{G_a(t')}\right) dt'$$

$$= -\int_0^t \frac{d}{dt'}\,[\ln G_a(t')]\,dt' = -\ln G_a(t),$$

and

$$\int_0^t \left[\frac{(1-p)\,f_b(t')\,dt'}{pF_a(t') + (1-p)\,F_b(t')}\right] \equiv \int_0^t \left(\frac{g_b(t')}{G_b(t')}\right) dt'$$

$$= -\int_0^t \frac{d}{dt'}\,[\ln G_b(t')]\,dt' = -\ln G_b(t).$$

Exponentiating the latter two identities between parallel and serial formulas yields (11) and (12). Q.E.D.

It is clear that solutions to the serial terms always exist. However, this is not always true for the parallel terms, where some p, f_a, f_b may give G_a, G_b that are not true survivor functions. Theorem 2 gives a necessary and sufficient condition and then, two easily applied sufficient conditions for well-defined solutions to G_a, G_b to exist. The sufficiency conditions are tantamount to ensuring that $pf_a(t)/(pF_a(t) + (1-p)\,F_b(t))$ and $(1-p)\,f_b(t)/(pF_a(t) + (1-p)\,F_b(t))$ both be hazard functions.

THEOREM 2 (Necessary and sufficient conditions). (A) *A necessary and sufficient condition for G_a, G_b in* (11) *and* (12) *to be survivor functions is that*

$$\int_0^\infty \left[\frac{pf_a(t')}{pF_a(t') + (1-p)\,F_b(t')}\right] dt' = +\infty, \tag{14}$$

$$\int_0^\infty \left[\frac{(1-p)\,f_b(t')}{pF_a(t') + (1-p)\,F_b(t')}\right] dt' = +\infty. \tag{15}$$

(B) *A sufficient condition that G_a, G_b be survivor functions is that*

$$\lim_{T\to+\infty}\left(\frac{F_b(T)}{F_a(T)}\right)=\lim_{T\to+\infty}\frac{\int_{t=T}^{+\infty}f_b(t)\,dt}{\int_{t=T}^{\infty}f_a(t)\,dt}=\alpha,\qquad 0<\alpha<\infty,$$

or by L'Hospital's rule,

$$\lim_{t\to+\infty}(f_b(t)/f_a(t))=\alpha.$$

(C) *Another sufficient condition that G_a, G_b be survivor functions is that*

$$\frac{pf_a(t)}{pF_a(t)+(1-p)\,F_b(t)}\qquad\text{and}\qquad\frac{(1-p)\,f_b(t)}{pF_a(t)+(1-p)\,F_b(t)}$$

both go to zero no faster than $a/(b+ct)$. That is, if $x(t)$ is either one of these then $(x(t)\div(a/(b+ct)))\to_{t\to+\infty}A$, $0<A\leqslant+\infty$.

Proof of (A). It is clearly necessary and sufficient that

$$\int_0^\infty\frac{pf_a(t')}{pF_a(t')+(1-p)\,F_b(t')}\,dt'=+\infty$$

and

$$\int_0^\infty\frac{(1-p)\,f_b(t')}{pF_a(t')+(1-p)\,F_b(t')}\,dt'=+\infty,$$

since then $G_a(t)$, $G_b(t)\to 0$, as $t\to+\infty$.

Proof of (B). Again the pertinent quantities under discussion are

$$I_a(T)=\int_{t=0}^{T}\frac{pf_a(t)}{pF_a(t)+(1-p)\,F_b(t)}\,dt$$

and

$$I_b(T)=\int_{t=0}^{T}\frac{(1-p)\,f_b(t)}{pF_a(t)+(1-p)\,F_b(t)}\,dt,$$

since it is necessary and sufficient that both of these expressions diverge as $T\to+\infty$ ((14), (15)).

Consider first $I_a(T)$. Since it is the tail behavior of $I_a(T)$ that is important to divergence of the integral we may attend to the limiting behavior of the integrand.

It is easily seen by the standard limit arguments that

$$\frac{pf_a(t)}{pF_a(t)+(1-p)\,F_b(t)}\xrightarrow[t\to+\infty]{}\frac{pf_a(t)}{pF_a(t)+(1-p)\,\alpha F_a(t)}$$
$$=\frac{p}{p+(1-p)\alpha}\cdot\frac{f_a(t)}{F_a(t)}=\frac{-p}{p+(1-p)\alpha}\,\frac{d\ln F_a(t)}{dt}.$$

That is, for large $T_1 < T_2$,

$$I_a(T_2) - I_a(T_1) \simeq \frac{-p}{p + (1-p)\alpha} \int_{T_1}^{T_2} \left(\frac{d \ln F_a(t)}{dt} \right) dt$$

$$= \frac{-p}{p + (1-p)\alpha} \ln \left[\frac{F_a(T_2)}{F_a(T_1)} \right].$$

Now, the significant part of this term for limiting behavior is

$$\ln F_a(T_2) \xrightarrow[T_2 \to +\infty]{} -\infty,$$

so that $I_a(T) \to +\infty$ and $G_a(T) = \exp(-I_a(T)) \to 0$ as desired.

Now consider

$$I_b(T) = \int_0^T \frac{(1-p) f_b(t')\, dt'}{pF_a(t') + (1-p) F_b(t')},$$

using the fact that $F_a(t) \to F_b(t)/\alpha$ to arrive at a similar conclusion for $G_b(t)$, namely, $G_b(t) \to 0$.

Proof of (C). The proof follows trivially from an elementary theorem of the integral calculus. Q.E.D.

After consideration of an example, a theorem will be proven showing that conditions (B) and (C) in Theorem 2 are independent in the sense that neither implies the other. Most of our examples involve the gamma distribution. This is because the gamma has been of significant interest to psychologists, because earlier work on parallel–serial equivalence emphasizes the general gamma distribution (e.g., see Townsend, in press) and because it is simple to manipulate.

EXAMPLE 1. Let

$$g_a(t) = \frac{(v_a t)^{k_a - 1}\, v_a e^{-v_a t}}{(k_a - 1)!}, \tag{16}$$

$$g_b(t) = \frac{(v_b t)^{k_b - 1}\, v_b e^{-v_b t}}{(k_b - 1)!}, \tag{17}$$

where k_a, k_b are arbitrarily fixed positive integers. That is, the parallel times are gamma distributed with k_a, k_b stages and v_a, v_b the respective rates for g_a, g_b. First find p:

$$p = \int_0^\infty g_a(t)\, G_b(t)\, dt = \int_0^\infty \left[\frac{(v_a t')^{k_a - 1}\, v_a e^{-v_a t'}}{(k_a - 1)!} \sum_{j=0}^{k_b - 1} \frac{(v_b t')^j e^{-v_b t'}}{(j)!} \right] dt' \tag{18}$$

and by an interchange of summation and integral,

$$p = \sum_{j=0}^{k_b-1} \left[\int_0^\infty \frac{v_a^{k_a} v_b^{\,j} e^{-(v_a+v_b)} t'^{k_a+j-1}}{(k_a-1)!\, j!}\, dt'\right]$$

$$= \sum_{j=0}^{k_b-1} \left[\frac{v_a^{k_a} v_b^{\,j} (k_a+j-1)!}{(k_a-1)!\, j! (v_a+v_b)^{k_a+j}}\right] \int_0^\infty \left[\frac{((v_a+v_b)\, t')^{k_a+j-1} (v_a+v_b)\, e^{\ (v_a+v_b)t'}}{(k_a+j-1)!}\, dt'\right]$$

$$= \sum_{j=0}^{k_b-1} s^{k_a} (1-s)^j \binom{k_a+j-1}{j} = \bar{p},$$

where $s = (v_a/(v_a + v_b))$. As can be seen above we use the symbol $\bar{p}$ to stand for the value of p written in terms of the parallel functions and parameters.

By symmetry we can conclude also that

$$1 - p = \sum_{j=0}^{k_a-1} s^j (1-s)^{k_b} \binom{k_b+j-1}{j} = 1 - \bar{p}.$$

Although it is not immediately obvious that adding the two sums yields a value of 1, such is indeed the case. Turning to $f_a(t)$ we find that

$$f_a(t) = \frac{1}{\bar{p}} \frac{(v_a t)^{k_a-1}\, v_a e^{-v_a t}}{(k_a-1)!} \sum_{j=0}^{k_b-1} \frac{(v_b t)^j\, e^{-v_b t}}{j!}, \tag{19}$$

and similarly

$$f_b(t) = \frac{1}{1-\bar{p}} \frac{(v_b t)^{k_b-1}\, v_b e^{-v_b t}}{(k_b-1)!} \sum_{j=0}^{k_a-1} \frac{(v_a t)^j\, e^{-v_a t}}{j!}, \tag{20}$$

and we have discovered two serial densities $f_a \neq f_b$, albeit curious ones, that are mathematically equivalent to the parallel expressions $g_a \cdot G_b$, $g_b \cdot G_a$, respectively; g_a, g_b being gamma densities with arbitrary rates and number of steps.

A special exponential case of this example is arrived at by setting $k_a = k_b = 1$. Then,

$$g_a(t) = v_a e^{-v_a t}, \qquad g_b(t) = v_b e^{-v_b t}$$

and

$$p = (v_a/(v_a + v_b)), \qquad f_a(t) = f_b(t) = (v_a + v_b)\, e^{-(v_a+v_b)t}.$$

Here, f_a, f_b are exponential, though in the more general case above, they were not gamma.

Now, continuing investigation of this example, we may turn matters around and inquire about the situation of (7) and (8) when p, f_a, and f_b are given by (18), (19), and (20), respectively. That is, we know a priori by (16) and (17) that G_a, G_b exist. Although we have not shown them expressed in terms of the serial terms we know a fortiori it is possible to do so. However, it is of interest to know whether (18), (19), and (20) satisfy conditions (B) and (C) in Theorem 2. We first prove that they satisfy condition (B).

Pursuing this goal, we note the possibility of directly examining f_b/f_a. However, since it might be useful to some investigations to have in hand the expressions for F_a, F_b from this example, we proceed with a proof based on these quantities. These are

$$F_a(t) = \int_{t'=t}^{\infty} f_a(t')\,dt' = \frac{1}{p}\int_{t'=t}^{\infty}\left[\frac{(v_a t')^{k_a-1}\, v_a e^{-v_a t'}}{(k_a-1)!}\sum_{j=0}^{k_b-1}\frac{(v_b t')^j\, e^{-v_b t'}}{j!}\right]dt'$$

$$= \frac{1}{p}\sum_{j=0}^{k_b-1}\frac{v_a^{k_a} v_b^{\,j}}{(v_a+v_b)^{k_a+j}}\cdot\frac{(k_a+j-1)!}{(k_a-1)!\,j!}$$

$$\cdot\int_{t'=t}^{\infty}\frac{(v_a+v_b)^{k_a+j}\, t'^{(k_a+j-1)} e^{-(v_a+v_b)t'}}{(k_a+j-1)!}\,dt'$$

$$= \frac{1}{p}\sum_{j=0}^{k_b-1}\left(\frac{v_a}{v_a+v_b}\right)^{k_a}\left(\frac{v_b}{v_a+v_b}\right)^{j}\binom{k_a+j-1}{j}\sum_{r=0}^{k_a+j-1}\frac{[(v_a+v_b)t]^r\, e^{-(v_a+v_b)t}}{r!}.$$

It is readily seen by symmetry that

$$F_b(t) = \frac{1}{1-p}\sum_{j=0}^{k_a-1}\left(\frac{v_b}{v_a+v_b}\right)^{k_b}\left(\frac{v_a}{v_a+v_b}\right)^{j}\binom{k_b+j-1}{j}$$

$$\cdot\sum_{r=0}^{k_b+j-1}\frac{[(v_a+v_b)t]^r\, e^{-(v_a+v_b)t}}{r!}.$$

Therefore, F_a, F_b are both binomially weighted series of partial Poisson sums. We ask if there exists an α such that

$$F_b(t)/F_a(t) \xrightarrow[t\to+\infty]{} \alpha.$$

In this ratio, the term $e^{-(v_a+v_b)t}$ cancels out and we are left with polynomials in t. Since the limit of the ratio then depends on the highest powers of t in F_a, F_b we calculate this together with the relevant coefficients. The highest power of t in F_a

occurs when $j = k_b - 1$ and in F_b when $j = k_a - 1$. It is easily seen that these are, respectively,

$$\frac{1}{\bar{p}}\left(\frac{v_a}{v_a+v_b}\right)^{k_a}\left(\frac{v_b}{v_a+v_b}\right)^{k_b-1}\binom{k_b+k_a-2}{k_b-1}\frac{(v_a+v_b)^{k_a+k_b-2}\,t^{k_a+k_b-2}}{(k_a+k_b-2)!}$$

$$= \frac{1}{\bar{p}}\,\frac{v_a^{k_a}v_b^{k_a-1}}{v_a+v_b}\cdot\frac{t^{k_a+k_b-2}}{(k_a-1)!\,(k_b-1)!}$$

and

$$\frac{1}{1-\bar{p}}\left(\frac{v_b}{v_a+v_b}\right)^{k_b}\left(\frac{v_a}{v_a+v_b}\right)^{k_a-1}\binom{k_a+k_b-2}{k_a-1}\frac{(v_a+v_b)^{k_b+k_a-2}\,t^{k_b+k_a-2}}{(k_b+k_a-2)!}$$

$$= \frac{1}{1-\bar{p}}\,\frac{v_b^{k_b}v_a^{k_a-1}}{(v_a+v_b)}\,\frac{t^{k_b+k_a-2}}{(k_b-1)!\,(k_a-1)!}.$$

That is, the highest power of t is identical for F_a, F_b and hence

$$\frac{F_b(t)}{F_a(t)} \xrightarrow[t\to+\infty]{} \frac{1-\bar{p}}{\bar{p}}\,\frac{v_b}{v_a} = \alpha,$$

so that α is equal to a ratio of the normalizing constants ($1-\bar{p}$ to $\bar{p}$) multiplied by a ratio of the g_b, g_a rates. This proves that the present example satisfies condition (B) in Theorem 2.

The proof that it also satisfies condition (C) is much simpler. Consider

$$\frac{pf_a(t)}{pF_a(t)+(1-p)\,F_b(t)} \doteq \frac{a}{b+ct}, \qquad \frac{(1-p)\,f_b(t)}{pF_a(t)+(1-p)\,F_b(t)} \doteq \frac{a}{b+ct}$$

and observe that the highest power of t in $f_a(t), f_b(t), F_a(t), F_b(t)$ is the same, namely, $t^{k_a+k_b-2}$ and that the exponential term $e^{-(v_a+v_b)t}$ cancels out. Here, both the above ratios approach functions of the form

$$b' + c't \to +\infty \qquad \text{as } t \to +\infty.$$

At this point it is pertinent to present Theorem 3, which proves the logical independence of conditions (B) and (C) in Theorem 2.

THEOREM 3. *Condition* (B) *of Theorem* 2 *is neither implied by, nor implies condition* (C).

(A). Condition (B) is not implied by (C). We show this by way of the following example.

EXAMPLE 2. Consider the serial densities and the corresponding survivor functions

$$f_a(t) = \frac{(\lambda t)^{k-1}\,\lambda e^{-\lambda t}}{(k-1)!}, \qquad F_a(t) = \sum_{j=0}^{k-1} \frac{(\lambda t)^j\, e^{-\lambda t}}{j!} \tag{21}$$

and

$$f_b(t) = \frac{(\lambda t)^{k}\,\lambda e^{-t}}{k!}, \qquad F_b(t) = \sum_{j=0}^{k} \frac{(\lambda t)^j\, e^{-\lambda t}}{j!}. \tag{22}$$

These serial functions satisfy condition (C) but not (B) of Theorem 2. Both (21) and (22) are gamma distributions with the same rate parameter but f_b supposes one more stage $(k+1)$ than does f_a.

Since Theorem 2 (B) implies that

$$\lim_{t\to+\infty} (F_b(t)/F_a(t)) = \alpha,$$

if this latter condition is violated, so is Theorem 2 (B).
But, (21) and (22) yield

$$\lim_{t\to+\infty} \frac{F_b(t)}{F_a(t)} = \lim_{t\to+\infty} \frac{(\lambda t)}{k} = +\infty$$

and condition (B) in Theorem 2 is therefore violated. However, viewing the integrals of interest I_a, I_b it is possible to show that they do satisfy Theorem 2 (C).

Taking $I_a(T)$ from (21) we find

$$I_a(T) = \int_0^T \Big[\frac{p(\lambda t')^{k-1}\,\lambda e^{-\lambda t'}[(k-1)!]^{-1}\,dt'}{p\sum_{j=0}^{k-1}((\lambda t')^j\, e^{-\lambda t'}/j!) + (1-p)\sum_{j=0}^{k}((\lambda t')^j\, e^{-\lambda t'}/j!)}\Big]$$
$$= \int_0^T \Big[\frac{p(\lambda t')^{k-1}\,\lambda[(k-1)!]^{-1}\,dt'}{\sum_{j=0}^{k-1}((\lambda t')^j/j!) + (1-p)((\lambda t')^k/k!)}\Big].$$

By a change of variable, this becomes

$$I_a(T) = \int_0^{\lambda T} \Big[\frac{px^{k-1}[(k-1)!]^{-1}\,dx}{\sum_{j=0}^{k-1}(x^j/j!) + (1-p)(x^k/k!)}\Big]$$
$$= \int_0^{\lambda T} \Big[\frac{[(k-1)!]^{-1}\,p\,dx}{\sum_{j=0}^{k-1}(x^{j-k+1}/j!) + (1-p)(x/k!)}\Big].$$

To scrutinize the tail behavior of $I_a(T)$, we view

$$\int_{\lambda T}^{\lambda T'} \frac{[(k-1)!]^{-1}\,p\,dx}{\sum_{j=0}^{k-1}(x^{j-k+1}/j!) + (1-p)(x/k!)}$$

and we note that for large T, T' this becomes close to

$$\int_{\lambda T}^{\lambda T'} \left[\frac{[(k-1)!]^{-1}\, p\, dx}{1+(1-p)(x/k!)}\right] = \frac{pk}{1-p}\cdot \log\left(\frac{T'}{T}\right).$$

That is, the integrand approaches a function of the form $a/(b+ct)$ and thus satisfies Theorem 2 (C). Considering next I_b of (22) we see that

$$\int_0^T \left(\frac{((1-p)(\lambda t')^k\, \lambda e^{-\lambda t}/k!)\, dt'}{p\sum_{j=0}^{k-1}((\lambda t')^j\, e^{-\lambda t}/j!) + (1-p)\sum_{j=0}^{k}((\lambda t')^j\, e^{-\lambda t}/j!)}\right)$$

$$= \int_0^{\lambda T} \frac{[k!]^{-1}\,(1-p)\, dx}{\sum_{j=0}^{k-1}(x^{j-k}/j!) + ((1-p)/k!)}$$

and since $x^{j-k} < 1$ for $j = 1, 2,\ldots, k-1$, for large x, it can be seen that the integrand approaches a constant and hence satisfies Theorem 2 (C).

B. Condition (C) is not implied by (B) in Theorem 2. Again an example will suffice to prove the result.

Example 3. Consider the survivor functions

$$F_a(t) = F_b(t) = F(t) = \log a/\log(t+a), \qquad 1 < a,$$

so that

$$f_a(t) = f_b(t) = f(t) = \log a/((t+a)\log^2(t+a))$$

and therefore,

$$\frac{f(t)}{F(t)} = \frac{1}{(t+a)\log(t+a)},$$

which converges to zero faster than $a/(b+ct)$, in violation of the condition (C) in Theorem 2. Yet

$$F_a(t)/F_b(t) \equiv 1,$$

trivially satisfying Theorem 2 (B).
Further,

$$G_a(t) = \exp\left[-p\int_0^t (f(t')/F(t'))\, dt'\right]$$

$$= \exp\left[-p\int_0^t (dt'/(t'+a)\log(t'+a))\right]$$

$$= \exp[\log(\log(t+a)/\log a)^{-p}]$$

$$= [\log a/\log(t+a)]^p,$$

and

$$G_b(t) = [\log a/\log(t + a)]^{1-p},$$

both well-defined survivor functions tending to zero as $t \to +\infty$. Q.E.D.

We now turn to the existence of f_a and f_b distributions that fail to yield acceptable solutions to the parallel functions.

An Example Failing to Satisfy the Functional Equations

It is perhaps mildly surprising that such an example may again be found within the gamma family of distributions.

Example 4. If

$$f_a(t) = \frac{(\lambda_a t)\, \lambda_a^{j-1} e^{-\lambda_a t}}{(j-1)!},$$

$$f_b(t) = \frac{(\lambda_b t)\, \lambda_b^{k-1} e^{-\lambda_b t}}{(k-1)!},$$

then it can be shown that $G_a(t)$ and $G_b(t)$ are well defined if and only if $\lambda_a = \lambda_b$ and $| k - j | \leqslant 1$.

First, without loss of generality it may be assumed that $k = j + m$, where m is some positive or zero valued integer. Now observe the integrands in the exponents of $G_a(t)$ and $G_b(t)$:

$$\frac{pf_a(t)}{pF_a(t) + (1-p)\, F_b(t)} = \frac{p((\lambda_a t)^{j-1}\, \lambda_a e^{-\lambda_a t}/(j-1)!)}{p \sum_{i=0}^{j-1} ((\lambda_a t)^i\, e^{-\lambda_a t}/i!) + (1-p) \sum_{i=0}^{k-1} ((\lambda_b t)^i\, e^{-\lambda_b t}/i!)}, \tag{23}$$

$$\frac{(1-p)\, f_b(t)}{pF_a(t) + (1-p)\, F_b(t)} = \frac{(1-p)((\lambda_b t)^{k-1}\, \lambda_b e^{-\lambda_b t}/(k-1)!)}{p \sum_{i=0}^{j-1} ((\lambda_a t)^i\, e^{-\lambda_a t}/i!) + (1-p) \sum_{i=0}^{k-1} ((\lambda_b t)^i\, e^{-\lambda_b t}/i!)}. \tag{24}$$

Dividing the numerators and denominators of (23) and (24) by $e^{-\lambda_a t}$, $e^{-\lambda_b t}$, respectively, we conclude that it is impossible that (23) and (24) can both give divergent integrals since the denominator in (23) contains $e^{-(\lambda_b - \lambda_a)t}$ and that in (24) contains $e^{-(\lambda_a - \lambda_b)t}$. Hence, unless $\lambda_a = \lambda_b$, one integrand diverges, yielding an appropriate $G(t)$ but the other converges too fast to result in a probabilistic G. Thus, it is necessary that $\lambda_a = \lambda_b$.

Letting $\lambda_a = \lambda_b$ (hence canceling out the exponentials), and again dividing the numerators and denominators of (23) and (24), this time by t^{j-1}, t^{k-1}, respectively, it becomes obvious that if $|k - j| \geqslant 2$ then one of the integrands will converge as t^{-2}, thus failing to properly define a G as a survivor function; although the other will be alright. On the other hand, if $|k - j| \leqslant 1$, then the greatest power of t possible in either one of the denominators is just 1, but these conditions then satisfy the sufficiency hypothesis of condition (C) of Theorem 2.

A special case of Example 4 not satisfying (7) and (8) is derived from the above by letting $k = j = 1$ but setting $\lambda_a \neq \lambda_b$. Hence, even the exponential distribution as constituting f_a, f_b violates (7) and (8) when the two serial rates are unequal. This is in accord with results established earlier but which confined the parallel distributions also to the exponential family (e.g., Townsend, 1972; in press).

We next discuss general parallel solutions when the serial densities are equivalent.

The Case When the Two Serial Densities Are Equivalent

A special situation of some interest arises when $f_a(t) \equiv f_b(t)$, as is expressed in the next theorem. A particular instance arose in Example 3 of Theorem 3.

THEOREM 4. *If $f_a(t) \equiv f_b(t)$ for all $t \geqslant 0$, then there always exist solutions to* (11) *and* (12) *of $G_a(t)$ and $G_b(t)$ and in terms of the serial survivor functions they are of the form*

$$G_a(t) = [F(t)]^p, \tag{25}$$

$$G_b(t) = [F(t)]^{1-p}. \tag{26}$$

Proof. From (11),

$$\begin{aligned} G_a(t) &= \exp\left[-\int_0^t \frac{pf(t')\,dt'}{pF(t') + (1-p)\,F(t')}\right] = \exp\left[-\int_0^t \frac{pf(t')}{F(t')}\,dt'\right] \\ &= \exp\left[-\int_0^t -p\left[\frac{d\ln F(t')}{dt'}\right]dt'\right] = \exp\left[p\int_0^{\ln F(t')} d\ln F(t')\right] \\ &= \exp[p\ln F(t)] = [F(t)]^p. \end{aligned}$$

In a similar fashion from (12), $G_b(t) = [F(t)]^{(1-p)}$. Q.E.D.

These results for $G_a(t)$ and $G_b(t)$ imply that $G_a(t)\cdot G_b(t) = F(t)$, which is itself a special case of the more general result, found by adding (7) to (8) and integrating

$$G_a(t)\cdot G_b(t) = pF_a(t) + (1-p)F_b(t).$$

There are some interesting properties associated with assuming $f_a(t) = f_b(t)$. One is that the conditional probability that a was completed first, given the (winning) time, is independent of that time. That is, knowing the time of the winner is knowing nothing with respect to which one finished first.

COROLLARY. *In either the parallel or serial case, the probability that a was first, given the winning time was t^*, is equal to the marginal probability that a was first.*

Proof. First writing the more general case, where $f_a(t) \not\equiv f_b(t)$, and letting $t^* =$ time of winning arrives at,

$$\begin{aligned} f(a \text{ first} \mid t^*) &= \frac{f(a \text{ first} \,\&\, t^*)}{f(t^*)} \\ &= \frac{p(a \text{ first})\, f(t^* \mid a \text{ first})}{f(t^*)} \\ &= \frac{p f_a(t^*)}{p f_a(t^*) + (1-p)\, f_b(t^*)}, \end{aligned}$$

and when $f_a(t) \equiv f_b(t)$ this reduces to p, the marginal probability that a is processed first. This is intuitively self-evident in the serial case, where the densities for a and b are equivalent to one another. It is somewhat less obvious in the case of the equivalent parallel equations. The comparable expression for the general parallel case is

$$f(a \text{ first} \mid t^*) = \frac{g_a(t^*)\, G_b(t^*)}{g_a(t^*)\, G_b(t^*) + g_b(t^*)\, G_a(t^*)}$$

which becomes, when the equivalence formulas are employed,

$$\begin{aligned} f(a \text{ first} \mid t^*) &= \frac{p[f(t^*)/[F(t^*)]^{1-p}][F(t^*)]^{1-p}}{p[f(t^*)/[F(t^*)]^{1-p}][F(t^*)]^{1-p} + (1-p)[f(t^*)/[F(t^*)]^{p}][F(t^*)]^{p}} \\ &= p \end{aligned}$$

as before. Q.E.D.

The fact that parallel–serial equivalence in this case implies

$$G_a(t) = [F(t)]^p, \qquad G_b(t) = [F(t)]^{1-p},$$

leads to the relationship between G_a and G_b of

$$G_a(t) = [G_b(t)]^{p/(1-p)}.$$

The above corollary may be interpreted as proving that this relationship between the G's is sufficient to obtain the lack of memory property. It is easy to show, and omitted here, that this power-function relationship is also necessary between two within-stage independent–parallel survivor functions for the lack of memory property to hold.

This lack of informational value is a generalization to other distributions of the exponential property in parallel processing where certainly

$$f(a \text{ first} \mid t^*) = \frac{v_a e^{-v_a t} e^{-v_b t}}{v_a e^{-v_a t} e^{-v_b t} + v_b e^{-v_b t} e^{-v_a t}}$$

$$= \frac{v_a}{v_a + v_b},$$

which of course corresponds to p in its serial counterpart.

It is also interesting to realize that the only nonexponential distribution capable of satisfying (7) and (8) adduced in an early paper (Townsend, 1972), the Weibull distribution, becomes a special case of (25) and (26) when the parameter constraints required for this purpose are employed.

Thus, when

$$f(t) = f_a(t) = f_b(t) = \alpha\rho(\rho t)^{\alpha-1}\, e^{-(\rho t)^\alpha}$$

and when

$$g_a(t) = \beta\tau(\tau t)^{\beta-1}\, e^{-(\tau t)^\beta}, \qquad g_b(t) = \gamma\omega(\omega t)^{\gamma-1}\, e^{-(\omega t)^\gamma},$$

to achieve parallel–serial equivalence it is sufficient that

$$p = \tau^\alpha/(\tau^\alpha + \omega^\alpha), \qquad \gamma = \alpha = \beta, \qquad \text{and } \rho^\alpha = \tau^\alpha + \omega^\alpha.$$

For then, looking at the germane survivor functions,

$$G_a(t) = e^{-(\tau t)^\beta} = e^{-(\tau t)^\alpha} = e^{-p(\tau^\alpha + \omega^\alpha)t^\alpha} = e^{-p(\rho t)^\alpha} = [F(t)]^p$$

and similarly

$$G_b(t) = [F(t)]^{(1-p)}.$$

Generalization to an Arbitrary Number of Elements

The development and solution of the parallel–serial functional equations for the minimum completion time for arbitrary n is quick. It results in solutions (28) and (30), corresponding to (9) and (10) and (11) and (12), respectively. The system of equations analogous to (7) and (8) may be expressed as

$$p_i f_i(t) = g_i(t) \prod_{\substack{j=1 \\ j \neq i}}^{n} G_j(t), \qquad i = 1, n. \tag{27}$$

For the serial solution in terms of the parallel terms, as before let

$$p_i = \int_{t'=0}^{\infty} g_i(t') \prod_{\substack{j=1 \\ j \neq i}}^{n} G_j(t')\, dt',$$

where p_i = probability of selecting element i to be processed first in the serial system, and then

$$f_i(t) = \frac{1}{p_i} g_i(t) \prod_{\substack{j=1 \\ j \neq i}}^{n} G_j(t). \tag{28}$$

To obtain the parallel solution when it exists in terms of the serial functions, begin by adding (27) over $i = 1, n$ and integrate, which yields

$$\sum_{i=1}^{n} p_i F_i(t) = \prod_{i=1}^{n} G_i(t). \tag{29}$$

Now divide the left- and right-hand sides of (27) by the left- and right-sides of (29), respectively, and then integrate again to reach

$$\ln G_i(t) = \int_0^t \frac{p_i f_i(t')}{\sum_{j=1}^{n} p_j F_j(t')}\, dt',$$

and exponentiating then achieves the desired result,

$$G_i(t) = \exp\left[-\int_0^t \frac{p_i f_i(t')}{\sum_{j=1}^{n} p_j F_j(t')}\, dt'\right], \qquad i = 1, n. \tag{30}$$

Conditions on the f_i to yield true solutions G_i (e.g., (14), (15)) are analogous to those developed for $n = 2$ and will not be pursued here. Also, examples may be constructed using principles established earlier for the case of $n = 2$.

It appears possible in some cases to push these developments to higher stages of processing, that is, to the distributions on intercompletion times between, say, the kth and $k + 1$th element to be completed. Under certain conditions this may be quite straightforward and for others, less so. Performing this for all stages $k = 1, n$ then would result in total parallel–serial equivalence on the completion of all n elements. Such questions are beyond the present scope but are intended for further investigation.

General Methods of Application and Related Problems

First some general ways in which the present equivalence results may be employed are presented. Then an area of current investigation will be indicated.

(1) As with other parallel–serial results, it may happen that one wishes to obtain predictions for, say, parallel models but they are quite complicated to derive from scratch. In such cases, it is sometimes feasible to derive the results using an equivalent serial model and then to use the equivalence mappings to obtain the desired parallel expressions.

(2) It is clear from results worked out above that the distributions constituting the equivalent "mimicking" model of the other type may be from a different family than that of the original type. In example (1) above, for instance, the parallel distributions were gamma, but the equivalent serial model was made up of distributions outside the gamma family. In fact, it is not difficult to show that f_a and f_b are weighted combinations (probability mixtures) of gamma distributions which are not, in general, themselves gamma. In some cases of investigation of serial versus parallel processing, it may be that the theorist is willing to posit a certain family of distributions. It may then be feasible to test parallelity against seriality. Thus, if one were to suppose processing were definitely gamma then predictions by a parallel model based on gamma distributions would have no serial equivalent with gamma densities except under special circumstances (as when the parallel densities are exponential), and empirical tests might then be constructed.

(3) Important issues in experimental psychology such as independence of element processing and the nature and relative limitations on processing capacity often have radically different interpretations in serial as opposed to parallel models. Such considerations enhance our understanding of how a variety of processing systems behave and can be important in providing intuitive and sometimes logical grounds for rejecting one model or interpretation in favor of another (see also Townsend, 1974).

In order to develop sensible tests of models based on opposing concepts, it is necessary to comprehend the regions of equivalence obtaining between such models. Thus, work to this end is naturally composed of two complementary branches. The one seeks to discover the broadest possible conditions that permit model–equivalence while the other attempts to uncover circumstances where the models are not equivalent and hence where potential empirical tests may be devised.

Some aspects of distinguishing models may come directly from the equivalence results themselves. Comment (2) above is an example of this as is the earlier finding that the minimum completion time can separate certain classes of parallel and serial models (e.g., exponential serial models with $\lambda_a \neq \lambda_b$ from all within-stage independent models).

Other distinctive characteristics of models under consideration may have to be invented. An example of the latter is the happy discovery that the assumption of different processing rates for matching of identical elements from matching of distinct elements, leads to a parallel–serial testing paradigm (PST: Townsend, in press) on virtually all serial models and all parallel models with exponential intercompletion times. One parallel–serial problem currently undergoing analysis is the generalization of the PST results to broader classes of parallel models such as those possessing within-stage independence.

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