CHAPTER 20

MATHEMATICAL PSYCHOLOGY

Trisha Van Zandt and James T. Townsend

Mathematical psychology is not, per se, a distinct branch of psychology. Indeed, mathematical psychologists can be found in any area of psychology. Rather, mathematical psychology characterizes the approach that mathematical psychologists take in their substantive domains. Mathematical psychologists are concerned primarily with developing theories and models of behavior that permit quantitative prediction of behavioral change under varying experimental conditions. There are as many mathematical approaches within psychology as there are substantive psychological domains. As with most theorists of any variety, the mathematical psychologist will typically start by considering the psychological phenomena and underlying structures or processes that she wishes to model.

A mathematical model or theory (and we do not distinguish between them here) is a set of mathematical structures, including a set of *linkage* statements. These statements relate variables, equations, and so on with components of the psychological process of interest and possibly also aspects of the stimuli or environment. Regardless of the domain, then, the first step in a mathematical approach is to quantify the variables, both independent and dependent, measured to study a psychological process. Quantification permits variables to be represented as parameters in a mathematical equation or statistical expression, the goal and defining feature of the mathematical psychology enterprise.

Mathematical psychologists, then, construct mathematical and statistical models of the processes they study. Some domains, such as vision, learning and memory, and judgment and decision making, which frequently measure easily quantifiable performance variables like accuracy and response time, exhibit a greater penetration of mathematical reasoning and a higher proportion of mathematical psychologists than other domains. Processes such as the behavior of individual neurons, information flow through visual pathways, evidence accumulation in decision making, and language production or development have all been subjected to a great deal of mathematical modeling. However, even problems like the dynamics of mental illness, problems falling in the domains of social or clinical psychology, have benefited from a mathematical modeling approach (e.g., see the special issue on modeling in clinical science in the *Journal of Mathematical Psychology* [Townsend & Neufeld, 2010]).

The power of the mathematical approach arises when unrealized implications of particular model structures become obvious after the mathematical representation of the model has been written down. By contrast, although verbal models might possess logical structure, the inability to interpret concepts in a mathematical fashion means that we cannot derive their logical implications. The ability to make such derivations for mathematical representations leads to better testability of theories, improved experimental designs targeting specific model predictions, and better data analyses—such analyses frequently being rooted in the statistical properties of the model variables.

This work was supported by National Science Foundation Grant BCS-0738059, National Institutes of Mental Health Grant 57717-04Al, and Air Force Office of Special Research Grant FA9550-07-1-0078.

DOI: 10.1037/13620-020
APA Handbook of Research Methods in Psychology: Vol. 2. Quantitative, Qualitative, Neuropsychological, and Biological, H. Cooper (Editor-in-Chief)

Mathematical modeling is the foundation of many of the physical sciences. In comparison to these, psychology is often described as a "young" science; as Laming (1973) described several decades ago, psychologists are still often focused on the questions of *what* is happening rather than *why* it is happening. Mathematical psychologists, pointing to the role that mathematics has played in the advancement of the physical sciences, have argued that advancement in psychology (and other social sciences) will depend on the extent to which mathematical theorizing is applied to psychological issues. A testament to this argument is the fact that although not all *important* psychological models are mathematical, a great many of them are.

Psychology differs from a physical science in more than its age, and the use of mathematical models will not, on its own, carry psychology forward. First, the systems studied by psychologists are far more complex than comparable systems in the physical sciences; and second, relationships between psychological variables are obscured by intrinsic variability in these complex systems. Thus, progress in psychology is tied to progress in statistics as well as technological developments that improve our ability to measure behavior. Even the best mathematical tools may not improve our understanding of some quirk of human behavior if we are unable to measure that behavior or discriminate between changes in that behavior and random fluctuations—fluctuations in either our measurements or the cognitive system we are studying.

The remainder of this chapter consists of three sections. The first outlines the history of mathematical psychology. The second describes its influence in modern experimental psychology (i.e., all those empirically driven and nonapplied fields of psychological study). The third discusses some ongoing issues in the field.

HISTORY

Foundations

Mathematical psychology traces its roots to before the beginning of experimental psychology, the latter usually dated from the 1879 establishment of Wilhelm Wundt's (1832–1920) laboratory in Leipzig, Germany. Eighteenth-century astronomers were well aware of the "personal equation" that characterized variations in observers' times to estimate when celestial objects moved past wires on a grid. These estimates were made with the assistance of a metronome. Thus, the estimates depended on the time the astronomer needed to refocus attention from the visual to the auditory modality. Clearly, the reliability of astronomical measurements were therefore heavily dependent on the degree to which observers differed from each other or, indeed, from one observation to the next.

Many astronomers were thus naturally concerned about precisely measuring the personal equation so that equipment could be appropriately recalibrated for different observers. Astronomer and mathematician Friedrich Bessel (1784–1846), however, was further interested in why such timing issues arose. He formulated a hypothesis that a second stimulus (whether the auditory click of the metronome or visual movement of the star) produced a disturbance in the perceptual system already processing the first stimulus (the visual movement of the star or the auditory click of the metronome; Duncombe, 1945). This was perhaps the first formalization of what was later to be known as the *psychological refractory period* (Rabbitt, 1969) or the *doctrine of prior entry* (Shore & Spence, 2005).

Psychophysics. Although the interesting question of the personal equation focused on the speed with which people can perform a task, a different historical branch began with how frequently people make different kinds of responses. Physiologist Ernst Weber (1795–1878) asked people to make yes–no judgments about whether the perceived weights of two objects were different. Holding the mass of the first object constant, he gradually increased the mass of the second object until people said "yes" ("different"). He was then able to define the *just noticeable difference*, the smallest increase in weight ΔI that a person could detect, and found that it was not a constant but instead a function of the weight I of the first object, or

$$\Delta I = kI. \tag{1}$$

Weber found that the value of k, which determined the just noticeable difference, was a constant for most values of I, establishing what we now refer to as *Weber's law*. This law holds for a wide range of intensities I and across different stimulus modalities.

Gustav Theodor Fechner (1801–1887), founder of the field of psychophysics and the first true mathematical psychologist, was inspired by Weber's work (Fechner, 1860/1889). Although trained as a physicist, Fechner yearned to solve one of philosophy's central and longstanding puzzles, namely, the relationship of the mind to the outside world and the physiological body itself. This giant of philosophical enigmas is known as the *mind–body problem*, which continues even now to attract attention from philosophers and cognitive scientists. Fechner tried to solve the mind–body problem by establishing a connection, via an equation, between events in the physical world and the psychological experience they evoked. In modern mathematical psychology, this problem is one of foundational measurement: How can psychological experience be quantified and related to physical intensity? Although Weber's work proposed a relationship between physical intensity and a person's *report* of their experience, Fechner sought a lawful and mathematical relationship between physical intensity and the experience itself.

Fechner (1860/1889) had the clever idea of employing Weber's law by making the assumption that the psychological experience of a just noticeable difference is the same for all values of I. That is, if the change in the psychological effect $\Delta S = c$ is equal to the same constant c for all just noticeable differences ΔI, then

$$\frac{\Delta S}{\Delta I} = \frac{c}{kI}, \tag{2}$$

or, in the limit,

$$dS = \frac{c}{kI}dI. \tag{3}$$

Applying the rules of calculus to solve this differential equation leads to the expression we now call *Fechner's law*: Psychological effects S are a logarithmic function of physical intensity I, or

$$S = K\log I, \tag{4}$$

for some constant K. Perhaps because of the slow decaying links with philosophy, no one thought at the time of experimentally testing the logarithm function prediction of Fechner's law. It was not until much later that Stevens (1957, 1961) tried and tested other formulas for the relation of sensation to stimulation.

Mental chronometry. While Weber and Fechner were laying the foundations of psychological measurement and psychophysics, Hermann von Helmholtz (1821–1894) was busy measuring the speed of nerve conduction in a frog's leg (Helmholtz, 1850). The realization that neural events take measureable time spurred F. C. Donders (1818–1889) to develop a system for mental chronometry, the measurement of the time required to perform cognitive tasks (Donders, 1868/1969). Donders asked people to perform three tasks involving two lights. Each task required three different cognitive components. We now refer to these tasks as *simple reactions* (respond when any light is perceived), *go–no go reactions* (respond when one specific light is perceived), and *choice reactions* (respond one way when one light is perceived, and a different way when the other light is perceived).

The cognitive components involved are perception, stimulus discrimination, and response selection. For simple reactions, only perception is required; for go–no go reactions, perception and stimulus discrimination are required; for choice reactions, perception, stimulus discrimination and response selection are required.

Donders (1868/1969) measured the response times for each task and then estimated the duration of the stimulus discrimination and response selection components by subtraction. The difference between simple reaction and go–no go reaction times gave an estimate of stimulus discrimination time. The difference between go–no go reaction and choice reaction times gave an estimate of response selection time. Donders's *method of subtraction* was the foundation of the idea, now fundamental in cognitive psychology, that differences in response time provide information about cognitive architecture—how the brain structures tasks to achieve different levels of performance. It has been used in a variety of experimental paradigms over the past 150 years and set the stage for such techniques of analysis as Sternberg's (1969) *additive factors method.*

Sternberg's (1969) approach proposed to determine whether two subprocesses involved in a psychological task were arranged in a strict series with one starting and finishing before the other (a serial process). Subsequent mathematical work extended the additive factors method in such a way that a very large class of potential mental architectures (including parallel processing in which task subprocesses are executed simultaneously) could also be directly tested (Schweickert, 1978; Schweickert & Townsend, 1989; Townsend, 1984).

Psychometrics. Experimental psychology took a sharp turn in 1914 with the publication of a landmark book by John B. Watson (1878–1958). This book heralded the dominance of the psychological school of behaviorism, which holds that behavior can be explained without reference to mental events. The school of behaviorism was beneficial to psychology by helping the nascent field break away from its sometimes murky philosophical roots. However, it relegated Fechner's (1860/1889) psychological measurement and Donders's (1868/1969) mental chronometry to the realm of pseudoscience and inhibited developments in the study of cognition for several decades. This did not entirely stop the growth of mathematical psychology as it was applied to behavior, however. In fact, one of the later so-called neobehaviorists, Clark Leonard Hull (1884–1952), used mathematics in his mission to form a general theory of learning and motivation (see Hull, 1952).

Applied concerns also required the development of psychologically motivated quantitative methods to solve problems in human engineering and measurement. The desire of colleges and the military to measure human intelligence and aptitude led to the rise of standardized testing and psychometrics just as the behaviorism movement was getting off the ground. Using tests to assess knowledge and aptitude has a history that extends back to ancient China (Elman, 2000). At the turn of the 20th century, the first intelligence tests were published (e.g., Binet, 1905/1916), and the College Entrance Examination Board (now the College Board) was founded, providing colleges and universities with a way to test fitness of applicants to complete their curriculum. Similarly, the military has always been concerned about fitting soldiers to jobs for which they are well-suited, and the demand for large-scale answers to problems of psychological measurement began at the beginning of World War I.

L. L. Thurstone (1887–1955), founder and first president of the Psychometric Society, made significant contributions to the theory of measurement and psychophysics. His work was concentrated on the problem of quantifying human ability—intelligence, primarily—and he worked closely with the Army and the Institute for Government Research writing civil service exams (Thurstone, 1952). His *Law of Comparative Judgment* (Thurstone, 1927) was the first work to establish the concept of a psychological continuum, a range of quantifiable psychological experience that could be used as the basis for psychophysical judgments. He later expanded this continuum to attitudes and ability, and it became the forerunner to the Bradley–Terry–Luce and Rasch models of psychometrics as well as signal-detection theory.

The Rise of Modern Mathematical Psychology

Modern mathematical psychology stems from three innovations in psychology and engineering: the first application of signal-detection theory to human performance (Swets, Tanner, & Birdsall, 1961), the application of information theory to encoding and decoding messages in the human cognitive system (Attneave, 1954), and two milestone publications in mathematical learning theory (Bush & Mosteller, 1955; Estes, 1950). Together these three areas of research laid the groundwork for the idea that remains central in cognitive psychology in the 21st century: The human being, as she makes her way through the world, operates like an information-processing device. Information from the external world is encoded by sensory transducers; this information is operated upon by various brain mechanisms to produce her perception of the world and to allow her to select appropriate responses to the world; finally, if necessary, she can activate her response effectors to manipulate the state of her external world.

Signal-detection theory, born from the problems of communications engineers during World War II, borrows its fundamentals from statistical decision theory. An observer is presented with a low-amplitude signal tone in a burst of white noise and must determine whether the signal is present. This stimulus gives rise to some sensory effect (perceived intensity), which varies randomly each time it is presented. This randomness is attributed to the inherent variability of sensory transduction or noise in the cognitive channel. Randomness means that signals (in which a tone is present) may sometimes have the same sensory effect as noise alone. Signal or noise decisions are made by evaluating either the likelihood that a particular sensory experience resulted from a signal or noise stimulus, or by evaluating the magnitude of the sensory effect relative to some minimum criterion required to call a stimulus a signal. The important contribution of signal-detection theory, which now forms the heart of many modern models of cognition, is that it provided a method for separating effects of response bias (how the likelihood or magnitude of experience is evaluated) from the discriminability of the stimulus.

Information theory also derived from work in statistics and communications engineering (see, e.g., Shannon & Weaver, 1949). It is a way of quantifying the information or uncertainty of a signal from the probabilities associated with each possible stimulus for that signal. Communications engineers were concerned with how signals could be compressed and how much information could be transmitted over a noisy communications channel; the analogy to the human decision maker was immediately obvious to psychologists (e.g., Attneave, 1954; Garner, 1974). Information theory not only could be used to quantify sets of stimuli and collections of responses but also could be used to measure how much information the cognitive system could transmit.

Information theory contributed to the "intelligent machine revolution," represented best perhaps by Wiener's influential 1948 book *Cybernetics; or, Control and Communication in the Animal and the Machine*. Cybernetics, the science of feedback control systems applied to biological systems, influenced our treatment of the human as an information processor but had its greatest impact on research in artificial intelligence. It also encouraged the application of general systems theory (and nonlinear dynamics) in cognitive modeling (see the section Neural Modeling).

From information theory came a tremendous amount of research exploring the processing limitations of humans, and this led to one of the first links between the dependent variables of response frequency and response time. The *Hick-Hyman law* of response time states that response time RT is a linear function of the amount of information H (measured in *bits*) transmitted through the system, or

$$RT = a + bH, \qquad (5)$$

where b is called the *channel capacity* of the human (Hick, 1952; Hyman, 1953). Later, Miller (1956) reviewed the channel capacity literature that encompassed a number of different tasks. In his classic paper "The Magic Number Seven Plus or Minus Two: Some Limits on Our Capacity for Processing Information," he argued that people were limited in their ability to process and transmit information to approximately 2.5 bits.

An outcome of Miller's (1956) work was the realization that information contained in a set of items might be less important than the size of the set itself. This, together with other work demonstrating that information theory did not provide a useful explanation for how information was processed (e.g., Leonard, 1959), arguably led to a decline in the use of information theory in cognitive modeling (see also Luce, 2003). However, it still remains a useful way to quantify psychological and behavioral concepts (e.g., Strange, Duggins, Penny, Dolan, & Friston, 2005). In addition, the general concept that humans can be studied as perceptual, cognitive, and action systems through which information flows led to the rise of the "information processing approach," which continues to dominate much of experimental psychology in the 21st century.

Signal detection and information theory both suggested ways that stimuli could be quantified. Furthermore, signal-detection theory suggested what a perceptual representation of stimuli might look like, pointing the way to a cognitive theory of stimulus discrimination. At this same time, new theories of learning were presented (Bush & Mosteller, 1955; Estes, 1950, 1957). Bush and Mosteller's (1955) work derived from the prevailing behavioristic view of animal learning. Their theories focused solely on changes in the observer's response probability over time. For example, Bush and Mosteller's approach employed a simple difference equation for learning. Consider a task in which an animal must learn to make one particular response. Letting $q(n)$ be the probability of an error on trial n, the simplest Bush and Mosteller model specified that $q(n) = aq(n - 1)$, where a is greater than 0 and less than 1. This means that the likelihood of an error decreases over trials—learning.

Consistent with behavioristic dogma, Bush and Mosteller's (1955) learning models did not speculate about the internal mental states of the observer. However, Estes's (1950, 1957) stimulus sampling theory, like signal-detection theory, diverged from this philosophy by representing stimuli as being composed of smaller "elements" that could be sampled and possibly conditioned (i.e., learned) by the observer (e.g., Atkinson & Estes, 1963). In contrast to Bush and Mosteller's approach, Estes's models made a large impact not only on research in learning, but also in memory. Many modern memory models have taken advantage of his conception of stimulus elements and the idea that stimulus elements become associated to various components of a task structure (e.g., Shiffrin & Steyvers, 1997).

The following decades saw the publication of several books that established mathematical psychology as a formal discipline. The first were the three volumes of the *Handbook of Mathematical Psychology* (Luce, Bush, & Galanter, 1963–1965a), followed by two volumes of *Readings in Mathematical Psychology* (Luce, Bush, & Galanter, 1963–1965b). These volumes were targeted primarily toward researchers active in the field. Atkinson, Bower, and Crothers published the more elementary *An Introduction to Mathematical Learning Theory* in 1966, but it was not until the 1970 publication of Coombs, Dawes, and Tversky's *Mathematical Psychology* that there existed an introductory textbook suitable for undergraduates. This text covered a broad set of topics, including signal detection, information, and learning theory as well as judgment and decisions, psychological measurement, and game theory. In 1973, Laming published a more advanced *Mathematical Psychology* text, but this text focused on models that could predict response times, a neglected domain in texts up until that time.

The *Journal of Mathematical Psychology* and the Society for Mathematical Psychology

By 1960, there were at least a large handful of truly mathematical psychologists. As Estes (2002) described, some of these psychologists regularly participated in what are now called the Social Science Research Council's Knowledge Institutions. These particular institutions were held at Stanford University for the purposes of training social scientists in mathematical and statistical techniques. In 1963, the idea was proposed to begin a new journal devoted to the publication of theoretical, mathematical articles in psychology; in 1964, the first issue of the *Journal of Mathematical Psychology* was published. Richard C. Atkinson, Robert R. Bush, Clyde H. Coombs, William K. Estes, R. Duncan Luce, William J. McGill, and George A. Miller served on the journal's first editorial board.

Several years later, mathematical psychologists began meeting informally in the summer to give papers and symposia. After a number of years, in 1976, the journal's editorial board organized the Society for Mathematical Psychology. Bylaws were drafted by Estes and Luce, together with William H. Batchelder and Bert F. Green; in 1977, the Society was formally incorporated. The Society has now, for more than 40 years, hosted an annual meeting each summer at which students and researchers from a wide range of disciplines have presented papers, posters, and symposia highlighting the application of mathematical and statistical models to problems in psychology, cognitive science, neuroscience, and cognitive engineering.

By the time the Society was getting under way in the United States, a similar organization had already been formed in Europe called the European Mathematical Psychology Group. The Group, although never formally incorporated, has met every year since it was founded by Jean-Claude Falmagne in 1971. The British Psychological Association began publishing the *British Journal of Mathematical and Statistical Psychology* in 1965, which was an offshoot of the *British Journal of Psychology: Statistical Section*

(1947–1952) and later the *British Journal of Statistical Psychology* (1953–1964). The papers appearing in the *British Journal* are from researchers in both psychometrics and mathematical psychology, and so it is in these pages that we can see most strongly the links between these two branches of quantitative psychology.

MODERN MATHEMATICAL PSYCHOLOGY

If one sampled a mathematical psychologist at random, one would find that she could be roughly categorized along four (nonorthogonal) dimensions. First of all, we might determine whether her modeling is strictly axiomatic or more loosely formulated. Next, we could determine whether she takes primarily a deterministic or a stochastic modeling approach. Then, we could ask whether her approach is primarily analytic or computational. Finally, her work may be primarily empirical or theoretical.

At the risk of oversimplification, an axiomatic approach is one in which the modeler writes down some primary definitions and then statements (axioms) about what should be true. For example, the modeler may specify mathematical definitions on the basis of the desire to represent situations in which people are presented with stimulus pairs and that their task is to choose the stimulus in the pair with the greatest perceived magnitude. An axiom might then be that, when presented with two tones (the stimulus pair), people should be able to identify correctly the one that is louder with probability greater than or equal to 0.5. These axioms, then, permit the association of mathematical variables and formulas to psychological concepts. Given a set of axioms, the modeler can go on to make logical inferences about what people should do under different conditions.

Axiomatic theorems do not usually address issues of intrinsic randomness—they tend to be deterministic. Given fixed-model parameters and a fixed stimulus, the model produces one and only one result. A stochastic model, by contrast, might produce very different results even when the parameters and the stimulus are fixed. Models of cognitive processing are frequently stochastic. Sequential sampling models, such as those reviewed by Ratcliff and Smith (2004), are a perfect example of the stochastic approach. Predictions about behavior are often focused on how dependent variables are distributed, and how the parameters of those distributions change with changes in experimental procedures.

An analytical approach is one in which dependent variables Y can be written as analytical expression involving independent variables X, or $Y = g(X)$ for a function g that does not require any messy numerical calculations (like taking a limit or integrating). The general linear model employed in regression is one example of an analytic expression. The expressions providing finishing time distributions for serial and parallel processing systems (e.g., Townsend, 1972, 1976; also see the section Model Testing, Evaluation, and Comparisons) are other examples.

In contrast, a nonanalytic expression does not allow one to write $Y = g(X)$ and generate predictions for Y algebraically; instead, a computer must be used to simulate the model or solve for Y. Often, the more complex the issue being addressed, the more likely it is that a computational approach will be necessary. Techniques for model comparison (Pitt, Myung, & Zhang, 2002), Bayesian model fitting (Lee, 2008), and models devoted to particularly intractable problems like text comprehension or language processing (e.g., Dennis & Kintsch, 2007) often require a computational approach.

Finally, many mathematical psychologists are also empiricists: They collect data to test their models. However, there is a subset of mathematical psychologists who rarely or never collect data; their work is primarily theoretical. When theoretical work suggests a certain empirical approach, they either collaborate with empiricists or, if it is available, they reanalyze already published data. These mathematical psychologists make theoretical contributions that suggest new mathematical representations of different psychological problems, or methodological contributions that provide new techniques of analysis. They are rather akin to theoretical physicists, some of whom had remarkable insights about the nature of things but were notoriously inept in the laboratory.

Foundational Measurement

Work in foundational measurement has followed the tradition established by Fechner (Falmagne,

APA-COOPER_V2-11-0601-020.indd 375 12/13/11 7:17:07 PM

1986). It is axiomatic, analytic, deterministic and, for the most part, theoretical. Its goal is to find measurement systems capable of quantifying psychological experience—to measure such experience. In the physical world, we measure objects frequently. We weigh ourselves, we compute distance, we mark time. Such physical quantities are based in extensive measurement, which requires the existence of a ratio scale (one with a true zero). We are so accustomed to making measurements of this sort that it seems natural to extend this kind of logic to psychological problems. However, the axioms of extensive measurement may not be justified for the measurement of psychological experience (cf. Narens, 1996).

Foundational measurement represents the first and oldest approach to applying mathematical reasoning to psychological problems. In many ways, foundational measurement set the tone for mathematical work in psychology, especially in psychophysics and decision making. The pioneering research of Patrick Suppes and R. Duncan Luce is especially notable. Suppes, although officially a philosopher, was perhaps the first, along with Dana Scott, to put a mathematical foundation under the psychological scales proposed by Stevens (1961; Scott & Suppes, 1958; Suppes & Zinnes, 1963). Luce brought the mathematics developed for foundational measurement to bear on problems both in psychophysics and decision making, leading to some of the field's most impressive contributions extending from the 1950s until the present day (Luce, 1959, 2004; Narens & Luce, 1986; Steingrimsson & Luce, 2005a, 2005b, 2006, 2007).

Psychophysics is amenable to a measurement approach because the physical quantity of interest is usually easy to measure (e.g., frequency of a tone) and there is a corresponding continuum of psychological experience (e.g., pitch). A fairly large body of beautiful mathematics has been developed to represent the psychological experience of magnitude in detection and discrimination tasks (e.g., Colonius & Dzhafarov, 2006; Falmagne, 1985; Krantz, Luce, Suppes, & Tversky, 1971; Luce, Krantz, Suppes, & Tversky, 1990; Suppes, Krantz, Luce, & Tversky, 1989).

For decision making, the goal of foundational measurement has been to derive scales of preference for objects on the basis of the frequency with which people choose one object over another. An axiomatic approach provides a basis for predicting what people should prefer in various circumstances. Violations of these predicted preferences point to incorrect axioms, which in turn leads to a greater understanding of how people make decisions. Tversky and Kahneman's work (e.g., Tversky & Kahneman, 1974, 1981) demonstrated above all that perfectly sensible axioms, such as those underlying expected utility theory, do not apply in many decision-making environments. Their work led to Kahneman's Nobel prize in Economics in 2002.

Work in foundational measurement is generally deterministic, meaning that it deals primarily with the algebraic properties of different measurement systems. This fact means that, although mathematically quite elegant, measurement theories are often quite removed from empirical treatments and, indeed, may be difficult or impossible to empirically evaluate because the variability of real data obscure and distort the relationships predicted by the theories (Luce, 2005; Narens & Luce, 1993). Although there have been several promising inroads to formulating stochastic approaches to foundational measurement over the past decade or so (Falmagne, Reggenwetter, & Grofman, 1997; Heyer & Niederée, 1989; Myung, Karabatsos, & Iverson, 2005), as yet there is no completely satisfactory solution.

Cognitive Modeling

Mathematical approaches to modeling cognitive processes are now fairly well ingrained in mainstream cognitive psychology. These approaches are equally balanced between analytic and computational models, but they are primarily stochastic and almost always empirical. It will not be possible for us to give a comprehensive treatment of every area in cognitive psychology for which mathematical modeling is important because this task would require many books. We focus on memory, categorization, choice response time, and neural modeling.

Memory. Nowhere else in experimental psychology has mathematical work had a greater impact than in the development of models for memory. Mathematical models of recognition and recall now set the standard for theoretical developments in this area and have driven empirical research before them. Memory models no longer follow the early examples set by statistical learning theory and models of information processing. It became obvious in the late 1960s and early 1970s that the complexity of the process to be modeled was not adequately captured by linearly decomposing it into a sequence of subtasks (e.g., Sternberg, 1966). This led to the development of connectionist models (see below) and machine-learning-inspired models that incorporate learning, problem solving, and language comprehension (e.g., Dennis, 2005; Jilk, Lebiere, O'Reilly, & Anderson, 2008; Kintsch, McNamara, Dennis, & Landauer, 2007).

Signal-detection theory still plays a very important role in most memory models. Older strength theories (Atkinson & Juola, 1973; Murdock, 1965; Parks, 1966) relied on the signal-detection framework as the basis for the old–new judgment. Newer global memory models—such as those proposed by Murdock (1982), Hintzman (1988), and Gillund and Shiffrin (1984), and even more recent models such as retrieving effectively from memory (Shiffrin & Steyvers, 1997)—develop encoding, storage, and retrieval architectures explaining how memory traces are established, maintained, and decay over time as well as how different memory traces become associated with each other and to the context in which they were experienced. Each of these models requires, however, an evaluation of memory strength for a recognition decision, and this evaluation is assumed to be performed within a signal-detection framework.

Although global memory models go some way toward explaining how memory strength contributes to recognition performance, many researchers have explored the contributions of other memory processes, often lumped together under the term *recall*. In this sense, recall is the ability to remember specific details of the remembered item, and this ability requires conscious effort. In contrast, recognition is based only on perceived strength, which happens effortlessly. Some memory work is focused on separating these different cognitive contributions to recognition decisions (e.g., Wixted, 2007). The receiver operating characteristic curve from signal detection is used to try and separate the signal-detection recognition component from the recall component. Dual-process memory theories thus combine the signal-detection approach with a less-quantitatively specified recall component.

Another theoretical avenue to multiprocess memory models are the multinomial processing-tree models explored by Batchelder and Riefer (1999). This general approach provides a way to explore many different structures producing categorical measurements of behavior. The multinomial processing tree model considers how different components of a task depend on each other (e.g., if recall fails, evaluate familiarity) but does not explain the mechanisms by which each component operates. So whereas signal-detection theory might explain the probability that a subject calls an item *old*, the multinomial approach only assumes that such a probability exists. The approach allows for a consideration of different latent structures and comparisons between different model architectures. It has been applied to a wide range of problems, most recently in the evaluation of cognitive deficits (e.g., Batchelder & Riefer, 2007). It lends itself well to Bayesian analysis and is closely linked to measurement problems in psychometrics (Batchelder, 2010).

Categorization. Categorization tasks ask observers to classify stimuli according to their types. These types may be quite concrete (e.g., chairs, dogs, diseases) or they may be very abstract. As in memory research, several influential mathematical models of categorization have set a standard for explanations of categorization behavior, and much of the empirical work in categorization over the past few decades has been driven by these models.

The first class of these models assumes that subjects construct a mental representation of different categories and that categorization decisions are made on the basis of the psychological distances (often referred to as similarities) between a stimulus and other objects (exemplars) in the mental space (Nosofsky, 1988; Nosofsky & Palmeri, 1997). These models take much inspiration from early work in multidimensional scaling (Torgerson, 1958), which was used to derive scales that could measure multidimensional stimuli and place them in relation to each other.

The second class of these models assumes that categories of stimuli can be represented as probability distributions in multidimensional space (Ashby, 1992; Ashby & Gott, 1988). Categorization judgments are made on the basis of a stimulus's location in that space relative to multidimensional discriminant functions (lines, planes, hyperplanes) that divide the space into categories. These models are called *decision-bound models*, and they are closely related to signal-detection models. They preserve the ideas of discriminability, bias, optimality, and so forth from signal detection, but the interest is more on how different stimulus dimensions are perceived and how those perceptions influence the placement of decision bounds.

Choice response time. Signal-detection theory also motivated most of the current, most successful mathematical models of simple choice, including Ratcliff's diffusion model (e.g., Ratcliff & Smith, 2004), Usher and McClelland's (2001) leaky competing accumulator model, the Poisson race model (Pike, 1973; Van Zandt, Colonius, & Proctor, 2000), Vickers's accumulator model (Smith & Vickers, 1988; Vickers, 1979), and (most recently) the linear ballistic accumulator (Brown & Heathcote, 2008). These models address how simple choices are made in most cognitive experiments. The theory from which all these *sequential sampling* models derive is quite simple: To make a decision, an observer engages a process of information gathering. Information is obtained by repeated sampling from the stimulus (if it is present) or from its mental representation (if it is not). Information is modeled as a continuum of sensory effect, and the stimulus representation from which information is sampled is provided by the signal-detection framework.

Each sample of information supports one of the two possible responses and is stored appropriately. The characteristics of this information (discrete or continuous), the time course of the sampling process (discrete or continuous), and the nature of the storage mechanisms (separate as in a race model or combined as in a random walk or diffusion) define the differences between the sequential sampling models. The important contribution of these models is their explanation of the speed–accuracy trade-off, an explanation that pulls the dependent variables of response time and frequency together within the same mechanism. To make a decision requires *enough* information—a threshold. If a decision must be made quickly, it must be made on the basis of less information, which will lead to less accurate decisions.

Not only do these models explain changes in both response time and response frequency but also the stochastic processes upon which they are based are (usually) simple enough that we can write down analytic expressions for the response time distributions and the response probabilities as functions of the parameters of the process. These models presently stand as the most successful explanations of response selection in simple tasks. We have some neurophysiological evidence that the brain uses neural modules as information collectors (Schall, 2003), which has encouraged continued application of these models across cognitive, clinical, and developmental psychology (Ratcliff, 2008; White, Ratcliff, Vasey, & McKoon, 2009).

In addition, these models are being brought to bear on classic problems in judgment and decision making (Busemeyer & Diederich, 2002; Merkle & Van Zandt, 2006; Pleskac & Busemeyer, 2010; Ratcliff & Starns, 2009; Van Zandt, 2000). In particular, the sequential sampling framework is being extended to judgments of confidence, leading to the simultaneous prediction of three dependent measures. This body of research, together with other models for judgment and decision making, has been named *cognitive decision theory*.

Neural modeling. One development of the 1980s was the advent of computational models inspired by neural processing mechanisms: parallel-distributed processing (McClelland & Rumelhart, 1986; Rumelhart & McClelland, 1986). The computational tools provided by connectionism have been widely applied to complex cognitive problems, such as speech and pattern recognition (e.g., Norris & McQueen, 2008), and are used in engineering applications including computer vision, handwriting recognition, textual analysis, and quality control.

There was a backlash in the late 1980s against the use of connectionist models for cognition, a backlash rooted in the argument that connectionist models were simply associationism (à la behaviorism) in disguise (Pinker & Mehler, 1988). Also, many cognitive psychologists argued that connectionist models, although they may provide good explanations of how the brain performs computations, do not necessarily make predictions about overt behavior (Fodor & Pylyshyn, 1988). Consequently, although neural modeling is an important and rapidly advancing enterprise, it does not look much like the cognitive connectionism of the early 1980s.

To model the brain well requires a deeper understanding of neuroanatomy than most cognitive psychologists possess, a set of skills that might include animal laboratory work that cognitive psychologists do not usually possess, and measuring devices (such as multiprobe electrode arrays and functional magnetic resonance imaging that were not available at the advent of connectionism. These deficiencies inspired new training programs designed to provide future researchers with these skills and to encourage collaboration between neuroscientists and behavioral scientists. There is now a huge body of research exploring neural models of cognition and brain function, models that are fundamentally quantitative in nature (e.g., Hasselmo, 2009; Howard & Kahana, 2002; O'Reilly & Frank, 2006), published in journals such as the *Journal of Computational Neuroscience* and *Neural Computation*.

At the time connectionist models became popular, there was a wave of enthusiasm for nonlinear dynamics as applied to problems in experimental psychology. This enthusiasm was driven not only by the obvious nonlinear dynamics of connectionist models, but also by *ecological psychology*, which is motivated by the idea that the human brain operates not only within the head but also within the environment (Gibson, 1950). The complex interactions between neural modules and the ever-changing external world can be modeled with general systems theory (Klir, 1969).

General systems theory encompasses the mathematics of catastrophe and chaos theory, which were the focus of much excitement and many symposia in the 1980s, but catastrophe and chaos theory never led to a revolution in mathematical cognitive modeling. The nonlinear dynamics approach, however, has led to an important bridge between mathematical biology and cognitive science, and to the focus on complex systems in psychology represented by the important work of Turvey (1990, 2009), Kelso (1995), and others (e.g., Large & Jones, 1999; Schmidt, Carello, & Turvey, 1990).

CURRENT ISSUES IN MATHEMATICAL MODELING

As mathematical psychology continues to mature, with the inevitable growing pains that process engenders, there has been some navel-gazing about where the discipline is headed (Luce, 1999, 2005; Townsend, 2008). In the heady 1950s and 1960s, mathematical psychology seemed the road toward a physical science of psychology, but perhaps the road did not go to the places the field's founders anticipated it would. If true, there might be several reasons for this, one being that (of course) one's children never grow up to become what one thought they would. Mathematical psychology prospers, even though it hasn't quite followed in its parents' footsteps.

Mathematical psychology is currently tackling two major issues, and both are focused primarily on methodology: How to distinguish between different models of the same process, and constructing Bayesian methods for the analysis of behavioral data. We discuss each of these before closing the chapter.

Model Testing, Evaluation, and Comparisons

One very important area in mathematical psychology addresses the problem of how to discriminate between different models. This is a long-standing problem in any field that constructs mathematical and statistical models, including statistics, where this issue is dealt with by considering issues of goodness of fit, variance accounted for, information criteria, Bayes factors, and so forth. In addition, the possibility that models based on very different psychological principles or mechanisms might be mathematically similar or even identical, the challenge of *model mimicking*, can generate a formidable threat to the uncovering of psychological laws. These and other important topics are outlined in this section.

APA-COOPER_V2-11-0601-020.indd 379 12/13/11 7:17:08 PM

Mathematical psychologists have recently focused on the issue of model complexity. That is, one model may fit data better than another not because it is a better model but only because it is more complex. Complexity is not just a question of how many parameters a model has. Two models may have the same number of parameters yet one of them (the more complex one) may be able to accommodate a wider range of data patterns than the other. Dealing with this issue borrows ideas from computer science and has its roots in information theory. Computer scientists have developed numerical techniques for quantifying complexity, opening the way for a different perspective on model selection. Pitt, Myung, and colleagues (Pitt, Kim, Navarro, & Myung, 2006; Pitt, Myung, Montenegro, & Pooley, 2008) are applying these techniques to a number of different problems, including the optimization of experimental designs for model testing and explorations of model parameter spaces.

Another method for model testing and selection is the powerful *state-trace analysis* methodology invented by Bamber (1979) and recently made popular by Dunn (2008). This technique is applied to problems for which the goal is to determine how many processes are contributing to the performance of a task (see the discussion of dual-process memory models). Many empirical pursuits try to answer the question of "how many processes" by looking for dissociations in patterns of data. That is, situations in which one experimental variable moves a dependent variable in the opposite direction (or not at all) of another variable. This finding is sometimes called *selective influence*, and it is used to argue that one variable affects one process whereas another variable affects a different process independent from the first. State trace analysis is a simple technique based on minimal assumptions. In particular, no particular probability distributions, other mathematical functions, or parameters are required. On the basis of this technique, Dunn and colleagues have argued that, in many situations, dissociations do not provide strong evidence for multiple processes (e.g., Dunn, 2004, 2008; Newell & Dunn, 2008).

Another approach to model testing uses the *strong inference* philosophy described by Platt (1964). The fundamental idea requires the scientist to set up a series of two or more juxtaposed hypotheses, rather than the more typical "there is a (predicted) effect" versus "there is no effect." For example, we might first test whether a psychological phenomenon takes place within short-term versus long-term memory and then follow that with a test of whether the coding system in that memory is verbal or spatial. Or, we might formulate two or more entire classes of models that obey contrasting fundamental principles. The scientist first tests among these models and, in a second stage of research, begins to test among more specific models within the *winning* class.

Research on serial versus parallel processing of elements in visual and memory search illustrates the challenges of model mimicking (e.g., Townsend, 1972, 1974) as well as the opportunity for implementation of strong inference (e.g., Townsend, 1984). For instance, parallel and serial models can, for some popular experimental designs, produce exactly the same predictions and thus be totally indiscriminable (e.g., Townsend, 1972). However, Townsend and Wenger (2004) presented mathematical formulations for large classes of parallel and serial models, formulations that highlight empirically distinguishable aspects of the different structures. They then use these class differences as assays to test the models. The strategies we mentioned earlier for identification of even more complex architectures (Schweickert, 1978; Schweickert & Townsend, 1989) also adhere to this strategy. With these assays, juxtaposed models can be refined to be more and more specific so that, for example, if the assays suggest that processing is parallel, then we might go on to test, say, a diffusion process (e.g., Ratcliff, 1978) versus a counting mechanism (e.g., Smith & Van Zandt, 2000).

The issue of how to select among different mathematical models of a process will never be considered "solved" any more than the perfect statistical procedure for all circumstances will be discovered. As models change over the years, techniques for testing and selecting them will necessarily evolve.

The Independent and Identically Distributed Problem and Bayesian Modeling

When subjects participate in a psychological experiment, they are usually asked to make more than one response. This is because one measurement does not allow the researcher to make inferences; intrinsic variability makes the measurement unreliable. A large number of responses from (usually) a large number of subjects across different conditions is collected to overcome this problem.

Although multiple observations solve the problem of statistical power, from a scientific perspective, they create another, entirely different problem. The measurement we obtain from a subject at one point in time is a function of all that has happened to that subject in the past. In particular, it is a function of the other measurements the subject has provided in our experiment. It is not possible to obtain repeated measurements under exactly the same conditions, even if the stimulus conditions remain exactly the same from trial to trial.

Nonetheless, we treat our data as independent and identically distributed (IID) observations from the same data-generating mechanism. Often, we assume the data are IID even if the observations are coming from different subjects. We blithely average, combine and collapse, even knowing that such operations can distort the shape of any underlying function relating independent to dependent variables (Estes & Maddox, 2005).

This is the IID problem, and it is presently being tackled by the application of hierarchical Bayesian modeling techniques to established processing models (Craigmile, Peruggia, & Van Zandt, 2011; Lee, 2008; Peruggia, Van Zandt, & Chen, 2002; Rouder & Lu, 2005; Rouder, Lu, Speckman, Sun, & Jiang, 2005). As in most Bayesian analyses, the goal is to determine the posterior distribution of some model parameters given a specified prior distribution and the model itself (the likelihood). In a hierarchical model, the parameters for each subject are assumed to be drawn from common hyperdistributions, so that the posterior hyperdistributions are informed by all the data from all the subjects. Thus, each subject's data are fit in a way that allows for individual differences, but inferences about effects of independent variables are made on the hyperparameter posteriors, which have "learned" from all the subjects' data combined.

Bayesian modeling has the potential to eliminate the problem of individual differences as well as order and other confounding effects (e.g., Craigmile et al., 2011), but it is a computationally difficult issue to address. There is currently a great deal of interest in treating response time data as time-series (e.g., Thornton & Gilden, 2005; Van Orden, Holden, & Turvey, 2005), an approach that recognizes that repeated observations from a single subject are correlated. At this time, however, the techniques usually employed for such analyses, as well as the conclusions that result from them, can be criticized (Wagenmakers, Farrell, & Ratcliff, 2004).

CONCLUSION

Modern mathematical psychology is a critical component of modern experimental psychology. From its earliest inception, mathematical psychology has made important contributions to our understanding of learning, memory, perception, and choice behavior; mathematical models continue to guide research in these areas as well as language acquisition and comprehension, problem solving, categorization, and judgment. Although modest in number, mathematical psychologists appear as leaders in many psychological disciplines, especially in cognition and neuroscience. They have been elected to the most esteemed societies in experimental psychology as well as the elite National Academy of Sciences. Several mathematical psychologists (Herbert Simon, Patrick Suppes, William K. Estes, and R. Duncan Luce) have received the highest scientific honor in the United States, that of receiving the National Medal of Science.

As experimental psychology matures, it is likely that our current definition for what constitutes mathematical psychology will change. Eventually, we hope, experimental psychologists will all use mathematical reasoning and develop mathematical models, and thus everyone will be mathematical psychologists under the definition we have provided in this chapter. However, just as there remain specifically mathematical subdisciplines in the physical and life sciences (e.g., physics, chemistry, and biology), we anticipate that mathematical psychology will endure as a unique endeavor among the different subdisciplines that make up the science of psychology.

References

Ashby, F. G. (Ed.). (1992). *Multidimensional models of perception and cognition*. Hillsdale, NJ: Erlbaum.

Ashby, F. G., & Gott, R. E. (1988). Decision rules in the perception and categorization of multidimensional stimuli. *Journal of Experimental Psychology: Learning, Memory, and Cognition*, *14*, 33–53. doi:10.1037/0278-7393.14.1.33

Atkinson, R. C., Bower, G. H., & Crothers, E. J. (1966). *An introduction to mathematical learning theory*. New York, NY: Wiley.

Atkinson, R. C., & Estes, W. K. (1963). Stimulus sampling theory. In R. D. Luce, R. R. Bush, & E. Galanter (Eds.), *Handbook of mathematical psychology* (Vol. 2, pp. 121–268). New York, NY: Wiley.

Atkinson, R. C., & Juola, J. F. (1973). Factors influencing the speed and accuracy of word recognition. In S. Kornblum (Ed.), *Attention and performance IV* (pp. 583–612). New York, NY: Academic Press.

Attneave, F. (1954). *Applications of information theory to psychology: A summary of basic concepts, methods, and results*. New York, NY: Holt, Rinehart & Winston.

Bamber, D. (1979). State-trace analysis: A method of testing simple theories of causation. *Journal of Mathematical Psychology*, *19*, 137–181. doi:10.1016/0022-2496(79)90016-6

Batchelder, W. H. (2010). Cognitive psychometrics: Using multinomial processing tree models as measurement tools. In S. E. Embretson (Ed.), *Measuring psychological constructs: Advances in model-based approaches* (pp. 71–93). Washington, D.C.: American Psychological Association. doi:10.1037/12074-004

Batchelder, W. H., & Riefer, D. M. (1999). Theoretical and empirical review of multinomial process tree modeling. *Psychonomic Bulletin and Review*, *6*, 57–86. doi:10.3758/BF03210812

Batchelder, W. H., & Riefer, D. M. (2007). Using multinomial processing tree models to measure cognitive deficits in clinical populations. In R. W. J. Neufeld (Ed.), *Advances in clinical cognitive science: Formal modeling of processes and symptoms* (pp. 19–50). Washington, D.C.: American Psychological Association. doi:10.1037/11556-001

Binet, A. (1916). New methods for the diagnosis of the intellectual level of subnormals. In E. S. Kite (Ed. & Trans.), *The development of intelligence in children*. Vineland, NJ: Publications of the Training School at Vineland. (Original work published 1905)

Brown, S. D., & Heathcote, A. (2008). The simplest complete model of choice reaction time: Linear ballistic accumulation. *Cognitive Psychology*, *57*, 153–178. doi:10.1016/j.cogpsych.2007.12.002

Busemeyer, J., & Diederich, A. (2002). Survey of decision field theory. *Mathematical Social Sciences*, *43*, 345–370. doi:10.1016/S0165-4896(02)00016-1

Bush, R. R., & Mosteller, F. (1955). *Stochastic models for learning*. New York, NY: Wiley.

Colonius, H., & Dzhafarov, E. N. (Eds.). (2006). *Measurement and representation of sensation*. Mahwah, NJ: Erlbaum.

Coombs, C. H., Dawes, R. M., & Tversky, A. (1970). *Mathematical psychology: An elementary introduction*. Englewood Cliffs, NJ: Prentice-Hall.

Craigmile, P. F., Peruggia, M., & Van Zandt, T. (2011). An autocorrelated mixture model for sequences of response time data. *Psychometrika, 75,* 613–632.

Dennis, S. (2005). A memory-based theory of verbal cognition. *Cognitive Science*, *29*, 145–193. doi:10.1207/s15516709cog0000_9

Dennis, S., & Kintsch, W. (2007). The text mapping and inference rule generation problems in text comprehension: Evaluating a memory-based account. In F. Schmalhofer & C. A. Perfetti (Eds.), *Higher level language processes in the brain: Inference and comprehension processes* (pp. 105–132). Mahwah, NJ: Erlbaum.

Donders, F. C. (1969). On the speed of mental processes (W. G. Koster, Trans.). *Acta Psychologica, 30*, 412–431. (Original work published 1868)

Duncombe, R. L. (1945). Personal equation in astronomy. *Popular Astronomy, 53*, 2–13,63–76, 110–121.

Dunn, J. C. (2004). Remember-know: A matter of confidence. *Psychological Review*, *111*, 524–542. doi:10.1037/0033-295X.111.2.524

Dunn, J. C. (2008). The dimensionality of the remember-know task: A state-trace analysis. *Psychological Review*, *115*, 426–446. doi:10.1037/0033-295-X.115.2.426

Elman, B. (2000). *A cultural history of civil examinations in late imperial China*. Berkeley: University of California Press.

Estes, W. K. (1950). Toward a statistical theory of learning. *Psychological Review*, *57*, 94–107. doi:10.1037/h0058559

Estes, W. K. (1957). Of models and men. *American Psychologist*, *12*, 609–617. doi:10.1037/h0046778

Estes, W. K. (2002). Traps in the route to models of memory and decision. *Psychonomic Bulletin and Review*, *9*, 3–25. doi:10.3758/BF03196254

Estes, W. K., & Maddox, W. T. (2005). Risks of drawing inferences about cognitive processes from model fits to individual versus average performance. *Psychonomic Bulletin and Review*, *12*, 403–408. doi:10.3758/BF03193784

Falmagne, J.-C. (1985). *Elements of psychophysical theory*. New York, NY: Oxford University Press.

Falmagne, J.-C. (1986). Psychophysical measurement and theory. In K. R. Boff, L. Kaufman, & J. P. Thomas

(Eds.), *Handbook of perception and human performance* (Vol. 1, Chapter 1). New York, NY: Wiley.

Falmagne, J.-C., Reggenwetter, M., & Grofman, B. (1997). A stochastic model for the evolution of preferences. In A. A. J. Marley (Ed.), *Choice, decision, and measurement: Essays in honor of R. Duncan Luce* (pp. 111–130). Mahwah, NJ: Erlbaum.

Fechner, G. T. (1889). *Elemente der psychophysik* [Elements of psychophysics] (W. M. Wundt, Ed.). Leipzig, Germany: Breitkopf & Härtel. (Original work published 1860)

Fodor, J. A., & Pylyshyn, Z. (1988). Connectionism and cognitive architecture: A critical analysis. *Cognition*, *28*, 3–71. doi:10.1016/0010-0277(88)90031-5

Garner, W. R. (1974). *The processing of information and structure*. Potomac, MD: Erlbaum.

Gibson, J. J. (1950). *The perception of the visual world*. Boston, MA: Houghton Mifflin.

Gillund, G., & Shiffrin, R. M. (1984). A retrieval model for both recognition and recall. *Psychological Review*, *91*, 1–67.

Hasselmo, M. E. (2009). A model of episodic memory: Mental time travel along encoded trajectories using grid cells. *Neurobiology of Learning and Memory*, *92*, 559–573. doi:10.1016/j.nlm.2009.07.005

Helmholtz, H. (1850). Vorläaufiger Bericht über die Fortpflanzungsgeschwindigkeit der nervenreizung [Preliminary report on the propagation velocity of nerve innervation]. *Archiv für Anatomie, Physiologie und Wissenschaftliche Medizin*, 71–73.

Heyer, D., & Niederée, R. (1989). Elements of a model-theoretic framework for probabilistic measurement. In E. E. Roskam (Ed.), *Mathematical psychology in progress* (pp. 99–112). Berlin, Germany: Springer.

Hick, W. E. (1952). On the rate of gain of information. *Quarterly Journal of Experimental Psychology*, *4*, 11–26. doi:10.1080/17470215208416600

Hintzman, D. L. (1988). Judgments of frequency and recognition memory in a multiple-trace memory model. *Psychological Review*, *95*, 528–551. doi:10.1037/0033-295X.95.4.528

Howard, M. W., & Kahana, M. J. (2002). A distributed representation of temporal context. *Journal of Mathematical Psychology*, *46*, 269–299. doi:10.1006/jmps.2001.1388

Hull, C. L. (1952). *A behavior system; an introduction to behavior theory concerning the individual organism*. New Haven, CT: Yale University Press.

Hyman, R. (1953). Stimulus information as a determinant of reaction time. *Journal of Experimental Psychology*, *45*, 188–196. doi:10.1037/h0056940

Jilk, D. J., Lebiere, C., O'Reilly, R. C., & Anderson, J. R. (2008). SAL: An explicitly pluralistic cognitive architecture. *Journal of Experimental and Theoretical Artificial Intelligence*, *20*, 197–218. doi:10.1080/09528130802319128

Kelso, S. J. A. (1995). *Dynamic patterns: The self-organization of brain and behavior*. Cambridge, MA: MIT Press.

Kintsch, W., McNamara, D. S., Dennis, S., & Landauer, T. K. (2007). LSA and meaning: In theory and application. In T. K. Landauer, D. S. McNamara, S. Dennis, & W. Kintsch (Eds.), *Handbook of latent semantic analysis* (pp. 467–479). Mahwah, NJ: Erlbaum.

Klir, G. J. (1969). *An approach to general systems theory*. New York, NY: Van Nostrand Reinhold.

Krantz, D. H., Luce, R. D., Suppes, P., & Tversky, A. (1971). *Foundations of measurement* (Vol. I). New York, NY: Academic Press.

Laming, D. R. (1973). *Mathematical psychology*. London, England: Academic Press.

Large, E. W., & Jones, M. R. (1999). The dynamics of attending: How people track time-varying events. *Psychological Review*, *106*, 119–159. doi:10.1037/0033-295X.106.1.119

Lee, M. D. (2008). Three case studies in the Bayesian analysis of cognitive models. *Psychonomic Bulletin and Review*, *15*, 1–15. doi:10.3758/PBR.15.1.1

Leonard, J. A. (1959). Tactile choice reactions. *Quarterly Journal of Experimental Psychology*, *11*, 76–83. doi:10.1080/17470215908416294

Luce, R. D. (1959). *Individual choice behavior: A theoretical analysis*. New York, NY: Wiley.

Luce, R. D. (1999). Where is mathematical modeling in psychology headed? *Theory and Psychology*, *9*, 723–737. doi:10.1177/0959354399096001

Luce, R. D. (2003). Whatever happened to information theory in psychology? *Review of General Psychology*, *7*, 183–188. doi:10.1037/1089-2680.7.2.183

Luce, R. D. (2004). Symmetric and asymmetric matching of joint presentations. *Psychological Review*, *111*, 446–454. doi:10.1037/0033-295X.111.2.446

Luce, R. D. (2005). An open measurement problem of interest. *Journal of Mathematical Psychology*, *49*, 440–442. doi:10.1016/j.jmp.2005.05.001

Luce, R. D., R. R. Bush, & E. Galanter (Eds.). (1963–1965a). *Handbook of mathematical psychology* (Vols. 1–2). New York, NY: Wiley.

Luce, R. D., R. R. Bush, & E. Galanter (Eds.). (1963–1965b). *Readings in mathematical psychology* (Vols. 1–2). New York, NY: Wiley.

Luce, R. D., Krantz, D. H., Suppes, P., & Tversky, A. (1990). *Foundations of measurement* (Vol. 3). New York, NY: Academic Press.

McClelland, J., & Rumelhart, D. (1986). *Parallel distributed processing: Explorations in the microstructure of* *cognition*: *Vol. 2. Psychological and biological models*. Cambridge, MA: MIT Press.

Merkle, E. C., & Van Zandt, T. (2006). An application of the Poisson race model to confidence calibration. *Journal of Experimental Psychology: General*, *135*, 391–408. doi:10.1037/0096-3445.135.3.391

Miller, G. A. (1956). The magic number seven plus or minus two: Some limits on our capacity for processing information. *Psychological Review*, *63*, 81–97. doi:10.1037/h0043158

Murdock, B. B. (1965). Signal detection theory and short-term memory. *Journal of Experimental Psychology*, *70*, 443–447. doi:10.1037/h0022543

Murdock, B. B. (1982). A theory for the storage and retrieval of item and associative information. *Psychological Review*, *89*, 609–626. doi:10.1037/0033-295X.89.6.609

Myung, J. I., Karabatsos, G., & Iverson, G. J. (2005). A Bayesian approach to testing decision making axioms. *Journal of Mathematical Psychology*, *49*, 205–225. doi:10.1016/j.jmp.2005.02.004

Narens, L. (1996). A theory of ratio magnitude estimation. *Journal of Mathematical Psychology*, *40*, 109–129. doi:10.1006/jmps.1996.0011

Narens, L., & Luce, R. D. (1986). Measurement: The theory of numerical assignments. *Psychological Bulletin*, *99*, 166–180. doi:10.1037/0033-2909.99.2.166

Narens, L., & Luce, R. D. (1993). Further comments on the "nonrevolution" arising from axiomatic measurement theory. *Psychological Science*, *4*, 127–130. doi:10.1111/j.1467-9280.1993.tb00475.x

Newell, B. R., & Dunn, J. C. (2008). Dimensions in data: Testing psychological models using state-trace analysis. *Trends in Cognitive Sciences*, *12*, 285–290. doi:10.1016/j.tics.2008.04.009

Norris, D., & McQueen, J. M. (2008). Shortlist B: A Bayesian model of continuous speech recognition. *Psychological Review*, *115*, 357–395. doi:10.1037/0033-295X.115.2.357

Nosofsky, R. M. (1988). Exemplar-based accounts of relations between classification, recognition, and typicality. *Journal of Experimental Psychology: Learning, Memory, and Cognition*, *14*, 700–708. doi:10.1037/0278-7393.14.4.700

Nosofsky, R. M., & Palmeri, T. (1997). Comparing exemplar-retrieval and decision-bound models of speeded perceptual classification. *Perception and Psychophysics*, *59*, 1027–1048. doi:10.3758/BF03205518

O'Reilly, R. C., & Frank, M. (2006). Making working memory work: A computational model of learning in the prefrontal cortex and basal ganglia. *Neural Computation*, *18*, 283–328. doi:10.1162/089976606775093909

Parks, T. E. (1966). Signal-detectability theory of recognition-memory performance. *Psychological Review*, *73*, 44–58. doi:10.1037/h0022662

Peruggia, M., Van Zandt, T., & Chen, M. (2002). Was it a car or a cat I saw? An analysis of response times for word recognition. In C. Gatsonis, R. E. Kass, A. Carriquiry, A. Gelman, D. Higdon, D. K. Pauler, & I. Verdinelli (Eds.), *Case studies in Bayesian statistics* (Vol. 6, pp. 319–334). New York, NY: Springer.

Pike, R. (1973). Response latency models for signal detection. *Psychological Review*, *80*, 53–68. doi:10.1037/h0033871

Pinker, S., & Mehler, J. (Eds.). (1988). *Connections and symbols*. Cambridge, MA: MIT Press.

Pitt, M. A., Kim, W., Navarro, D. J., & Myung, J. I. (2006). Global model analysis by parameter space partitioning. *Psychological Review*, *113*, 57–83. doi:10.1037/0033-295X.113.1.57

Pitt, M. A., Myung, I. J., & Zhang, S. (2002). Toward a method of selecting among computational models of cognition. *Psychological Review*, *109*, 472–491. doi:10.1037/0033-295X.109.3.472

Pitt, M. A., Myung, J. I., Montenegro, M., & Pooley, J. (2008). Measuring model flexibility with parameter space partitioning: An introduction and application example. *Cognitive Science: A Multidisciplinary Journal*, 1285–1303.

Platt, J. R. (1964). Strong inference. *Science, 146*, 347–353.

Pleskac, T. J., & Busemeyer, J. R. (2010). *Two stage dynamic signal detection theory: A dynamic and stochastic theory of confidence, choice, and response time.* Manuscript submitted for publication.

Rabbitt, P. (1969). Psychological refractory delay and response-stimulus interval duration in serial, choice-response tasks. *Acta Psychologica*, *30*, 195–219. doi:10.1016/0001-6918(69)90051-1

Ratcliff, R. (1978). A theory of memory retrieval. *Psychological Review, 85*, 59–108.

Ratcliff, R. (2008). Modeling aging effects on two-choice tasks: Response signal and response time data. *Psychology and Aging*, *23*, 900–916. doi:10.1037/a0013930

Ratcliff, R., & Smith, P. L. (2004). A comparison of sequential sampling models for two-choice reaction time. *Psychological Review*, *111*, 333–367. doi:10.1037/0033-295X.111.2.333

Ratcliff, R., & Starns, J. (2009). Modeling confidence and response time in recognition memory. *Psychological Review*, *116*, 59–83. doi:10.1037/a0014086

Rouder, J. N., & Lu, J. (2005). An introduction to Bayesian hierarchical models with an application in the theory of signal detection. *Psychonomic Bulletin and Review*, *12*, 573–604. doi:10.3758/BF03196750

Rouder, J. N., Lu, J., Speckman, P., Sun, D., & Jiang, Y. (2005). A hierarchical model for estimating response time distributions. *Psychonomic Bulletin and Review*, *12*, 195–223.

Rumelhart, D., & McClelland, J. (1986). *Parallel distributed processing: Explorations in the microstructure of cognition*: *Vol. 1. Foundations*. Cambridge, MA: MIT Press.

Schall, J. D. (2003). Neural correlates of decision processes: neural and mental chronometry. *Current Opinion in Neurobiology*, *13*, 182–186. doi:10.1016/S0959-4388(03)00039-4

Schmidt, R. C., Carello, C., & Turvey, M. T. (1990). Phase transitions and critical fluctuations in the visual coordination of rhythmic movements between people. *Journal of Experimental Psychology: Human Perception and Performance*, *16*, 227–247. doi:10.1037/0096-1523.16.2.227

Schweickert, R. (1978). A critical path generalization of the additive factor methods analysis of a Stroop task. *Journal of Mathematical Psychology*, *18*, 105–139. doi:10.1016/0022-2496(78)90059-7

Schweickert, R., & Townsend, J. T. (1989). A trichotomy method: Interactions of factors prolonging sequential and concurrent mental processes in the stochastic PERT networks. *Journal of Mathematical Psychology*, *33*, 328–347. doi:10.1016/0022-2496(89)90013-8

Scott, D. S., & Suppes, P. (1958). Foundational aspects of theories of measurement. *Journal of Symbolic Logic*, *23*, 113–128. doi:10.2307/2964389

Shannon, C. E., & Weaver, W. (1949). *The mathematical theory of communication*. Urbana: University of Illinois Press.

Shiffrin, R. M., & Steyvers, M. (1997). A model for recognition memory: REM—retrieving effectively from memory. *Psychonomic Bulletin and Review*, *4*, 145–166. doi:10.3758/BF03209391

Shore, D. I., & Spence, C. (2005). Prior entry. In L. Itti, G. Rees, & J. K. Tsotsos (Eds.), *Neurobiology of attention* (pp. 89–95). New York, NY: Elsevier. doi:10.1016/B978-012375731-9/50019-7

Smith, P. L., & Van Zandt, T. (2000). Time-dependent Poisson counter models of response latency in simple judgment. *British Journal of Mathematical and Statistical Psychology, 53*, 293–315.

Smith, P. L., & Vickers, D. (1988). The accumulator model of two-choice discrimination. *Journal of Mathematical Psychology*, *32*, 135–168. doi:10.1016/0022-2496(88)90043-0

Steingrimsson, R., & Luce, R. D. (2005a). Evaluating a model of global psychophysical judgments: I. Behavioral properties of summations and productions. *Journal of Mathematical Psychology*, *49*, 290–307. doi:10.1016/j.jmp.2005.03.003

Steingrimsson, R., & Luce, R. D. (2005b). Evaluating a model of global psychophysical judgments: II. Behavioral properties linking summations and productions. *Journal of Mathematical Psychology*, *49*, 308–319. doi:10.1016/j.jmp.2005.03.001

Steingrimsson, R., & Luce, R. D. (2006). Empirical evaluation of a model of global psychophysical judgments: III. A form for the psychophysical function and intensity filtering. *Journal of Mathematical Psychology*, *50*, 15–29. doi:10.1016/j.jmp.2005.11.005

Steingrimsson, R., & Luce, R. D. (2007). Empirical evaluation of a model of global psychophysical judgments: IV. Forms for the weighting function. *Journal of Mathematical Psychology*, *51*, 29–44. doi:10.1016/j.jmp.2006.08.001

Sternberg, S. (1966). High-speed scanning in human memory. *Science, 153*, 652–654.

Sternberg, S. (1969). The discovery of processing stages: Extensions of Donders's method. In W. G. Koster (Ed.), *Attention and performance II* (pp. 276–315). Amsterdam, the Netherlands: North-Holland.

Stevens, S. S. (1957). On the psychophysical law. *Psychological Review, 64*, 153–181.

Stevens, S. S. (1961). To honor Fechner and repeal his law. *Science*, *133*, 80–86. doi:10.1126/science.133.3446.80

Strange, B. A., Duggins, A., Penny, W., Dolan, R. J., & Friston, K. J. (2005). Information theory, novelty and hippocampal responses: Unpredicted or unpredictable? *Neural Networks*, *18*, 225–230. doi:10.1016/j.neunet.2004.12.004

Suppes, P., Krantz, D. H., Luce, R. D., & Tversky, A. (1989). *Foundations of measurement* (Vol. II). New York, NY: Academic Press.

Suppes, P., & Zinnes, J. (1963). Basic measurement theory. In R. D. Luce, R. R. Bush, & E. Galanter (Eds.), *Handbook of mathematical psychology* (pp. 1–76). New York, NY: Wiley.

Swets, J., Tanner, W. P., & Birdsall, T. G. (1961). Decision processes in perception. *Psychological Review*, *68*, 301–340. doi:10.1037/h0040547

Thornton, T. L., & Gilden, D. L. (2005). Provenance of correlations in psychological data. *Psychonomic Bulletin and Review*, *12*, 409–441. doi:10.3758/BF03193785

Thurstone, L. L. (1927). A law of comparative judgement. *Psychological Review*, *34*, 273–286. doi:10.1037/h0070288

Thurstone, L. L. (1952). L. L. Thurstone. In E. G. Boring, H. S. Langfeld, & H. W. R. M. Yerkes (Eds.), *A history of psychology in autobiography* (Vol. IV, pp. 295–321). Worcester, MA: Clark University Press. doi:10.1037/11154-014

Torgerson, W. S. (1958). *Theory and methods of scaling*. New York, NY: Wiley.

Townsend, J. T. (1972). Some results concerning the identifiability of parallel and serial processes. *British Journal of Mathematical and Statistical Psychology*, *25*, 168–199.

Townsend, J. T. (1974). Issues and models concerning the processing of a finite number of inputs. In B. H. Kantowitz (Ed.), *Human information processing: Tutorials in performance and cognition* (pp. 133–168). Hillsdale, NJ: Erlbaum.

Townsend, J. T. (1976). Serial and within-stage independent parallel model equivalence on the minimum completion time. *Journal of Mathematical Psychology*, *14*, 219–238. doi:10.1016/0022-2496(76)90003-1

Townsend, J. T. (1984). Uncovering mental processes with factorial experiments. *Journal of Mathematical Psychology*, *28*, 363–400. doi:10.1016/0022-2496(84)90007-5

Townsend, J. T. (2008). Mathematical psychology: Prospects for the 21st century. *Journal of Mathematical Psychology*, *52*, 269–280. doi:10.1016/j.jmp.2008.05.001

Townsend, J. T., & Neufeld, R. J. (Eds.). (2010). Contributions of mathematical psychology to clinical science and assessment [Special issue]. *Journal of Mathematical Psychology*, *54*(1).

Townsend, J. T., & Wenger, M. J. (2004). The serial-parallel dilemma: A case study in a linkage of theory and method. *Psychonomic Bulletin and Review*, *11*, 391–418. doi:10.3758/BF03196588

Turvey, M. T. (1990). Coordination. *American Psychologist*, *45*, 938–953.

Turvey, M. T. (2009). On the notion and implications of organism-environment system: Introduction. *Ecological Psychology*, *21*, 97–111. doi:10.1080/10407410902877041

Tversky, A., & Kahneman, D. (1974). Judgment under uncertainty: Heuristics and biases. *Science*, *185*, 1124–1131. doi:10.1126/science.185.4157.1124

Tversky, A., & Kahneman, D. (1981). The framing of decisions and the psychology of choice. *Science*, *211*, 453–458. doi:10.1126/science.7455683

Usher, M., & McClelland, J. L. (2001). On the time course of perpetual choice: The leaky competing accumulator model. *Psychological Review*, *108*, 550–592. doi:10.1037/0033-295X.108.3.550

Van Orden, G. C., Holden, J. G., & Turvey, M. T. (2005). Human cognition and 1/*f* scaling. *Journal of Experimental Psychology: General*, *134*, 117–123. doi:10.1037/0096-3445.134.1.117

Van Zandt, T. (2000). ROC curves and confidence judgments in recognition memory. *Journal of Experimental Psychology: Learning, Memory, and Cognition*, *26*, 582–600. doi:10.1037/0278-7393.26.3.582

Van Zandt, T., Colonius, H., & Proctor, R. W. (2000). A comparison of two response time models applied to perceptual matching. *Psychonomic Bulletin and Review*, *7*, 208–256. doi:10.3758/BF03212980

Vickers, D. (1979). *Decision processes in visual perception*. New York, NY: Academic Press.

Wagenmakers, E.-J., Farrell, S., & Ratcliff, R. (2004). Estimation and interpretation of 1/f α noise in human cognition. *Psychonomic Bulletin and Review, 11*, 579–615.

Watson, J. B. (1914). *Behavior: An introduction to comparative psychology*. New York, NY: Henry Holt.

White, C., Ratcliff, R., Vasey, M., & McKoon, G. (2009). Dysphoria and memory for emotional material: A diffusion model analysis. *Cognition and Emotion, 23*, 181–205

Wiener, N. (1948). *Cybernetics; or, control and communication in the animal and the machine.* Cambridge, MA: MIT Press.

Wixted, J. T. (2007). Dual-process theory and signal-detection theory of recognition memory. *Psychological Review*, *114*, 152–176. doi:10.1037/0033-295X.114.1.152